Title: Electronic coherence and electronic coherent states

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Abstract: Quantum Information Workshop

Fermionic coherence (and clocks)

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- In quantum optics, a coherent state of light can operate as a time reference, i.e. a clock
- Within an energy superselection framework, the entangled coherent state representation admits clock states that are locked to evolutions of other fields
- What about clocks for fermions?
- We consider coherent states and operational coherence for fermions

[1] Institute of Theoretical Physics, Masaryk University Brno, Czech Republic

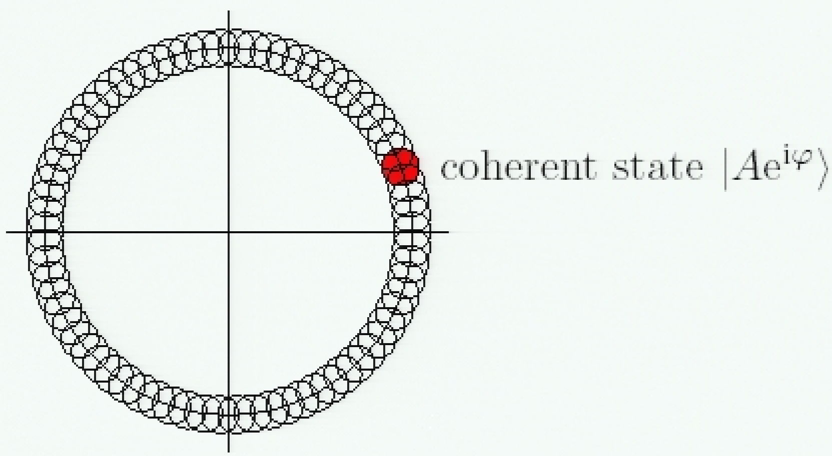
Number vs coherent-state representation

- Based on Sanders, Bartlett, Rudolph & Knight, Phys. Rev. A **68**, 042329 (2003)]
- Photon fields at optical frequencies are diagonal in the number state basis:

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|$$

- Express number state as a superposition of coherent states on a circle:

$$|n\rangle = \frac{1}{\sqrt{\mathrm{e}^{-m} m^n / n!}} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \, \mathrm{e}^{-\mathrm{i}n\varphi} |\sqrt{m}\,\mathrm{e}^{\mathrm{i}\varphi}\rangle$$



Superselection and entangled coherent states

- If the entire universe has a definite charge or definite L_z (z-component of angular momentum), then the density matrix of any subsystem is diagonal in charge or L_z, respectively [Y. Aharonov and L. Susskind, Phys. Rev. **155**, 1428 (1967)]
- Even with charge or energy superselection, coherence is permitted via entangled coherent state representation
- Coherent-state representation of the a two-mode number state

$$|n\rangle_1|n\rangle_2 = \frac{1}{\mathrm{e}^{-m}m^n/n!}\int\frac{\mathrm{d}\varphi_1}{2\pi}\frac{\mathrm{d}\varphi_2}{2\pi}\,\mathrm{e}^{-\mathrm{i}n(\varphi_1+\varphi_2)}|\sqrt{m}\,\mathrm{e}^{\mathrm{i}\varphi_1}\rangle|\sqrt{m}\,\mathrm{e}^{\mathrm{i}\varphi_2}\rangle$$

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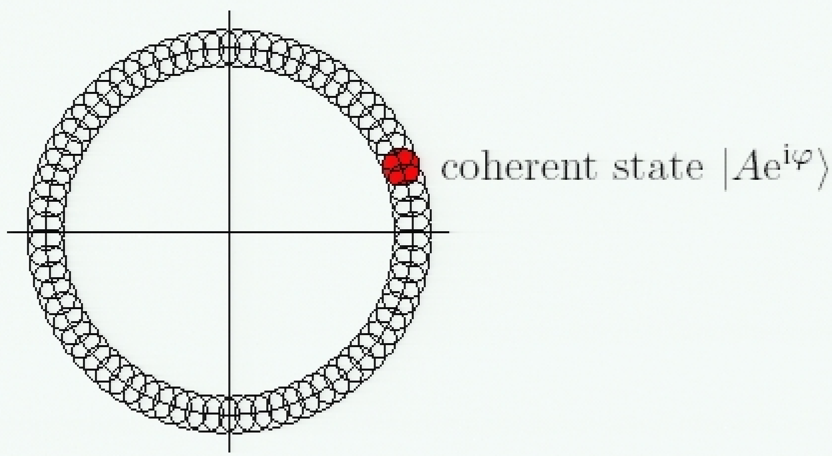

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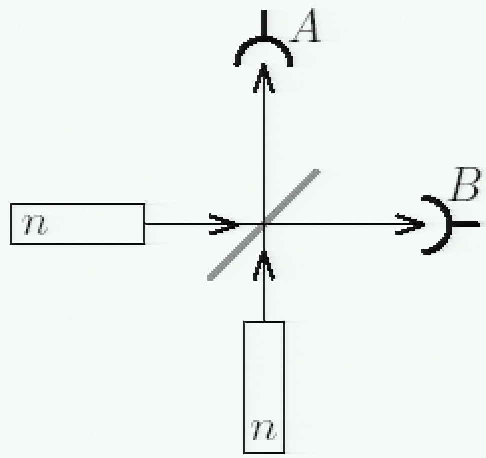
Example: interference of two number states

- Experiment: two number states in leaky cavities, output modes combined on beam splitter, photon numbers A and B are counted



- After detection, the original state $|n\rangle_1|n\rangle_2$ transforms into

$$\frac{1}{e^{-m}m^n/n!}\int \frac{d\varphi_1}{2\pi}\frac{d\varphi_2}{2\pi}\, e^{-in(\varphi_1+\varphi_2)}\, C_{A,B}(\varphi_1,\varphi_2) \times |\sqrt{(1-\epsilon)m}\, e^{i\varphi_1}\rangle|\sqrt{(1-\epsilon)m}\, e^{i\varphi_2}\rangle$$

- $|C_{A,B}(\varphi_1,\varphi_2)|$ depends only on $\varphi_1-\varphi_2$ and is peaked at some value $\varphi_1-\varphi_2=\Delta$
- The states in the cavities become entangled: their relative phase is no longer uncertain – phase locking

Photonic coherence

- Operational coherence - correlation functions
- Two conditions: (i) weak coherence and (ii) strong coherence.

  (i) normalized correlation functions are unity

  (ii) correlation functions factorize for all orders over the entire spatial domain
- Coherent states of light:

  Eigenstates of the annihilation operator

  When transformed on a beam splitter (with vacuum on the other port), remain unentangled
- Correlation functions of all orders factorize for them
- No means to produce coherent states at optical frequencies at the present time

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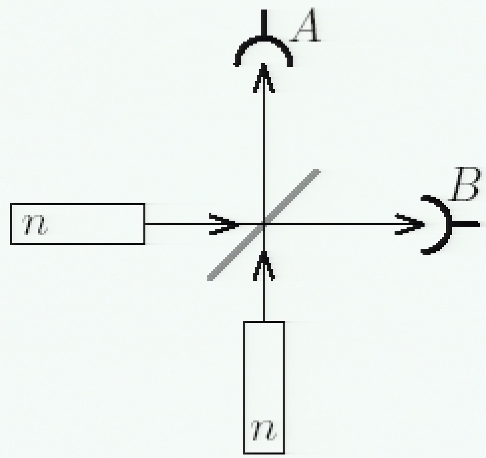

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Electronic coherence - Bartlett, Sanders, Oliver, Yamamoto

- Consider coherent electron beam; we need some measure of coherence
- Coherence (or correlation) functions – relate field at different points
- Analogy with well-known correlation functions for light
- Current correlation functions – conserve particle number (advantage)
- Raising $\hat{J}^{(-)}$ and lowering $\hat{J}^{(+)}$ current operators are respectively negative- and positive-frequency parts of the current operator

$$\hat{J}(\mathbf{r},t) = \sum_{k,k'} f_{kk'}(\mathbf{r})\hat{a}^\dagger_{k'}\hat{a}_k \quad \text{with } f_{kk'} = \frac{\mathrm{i}e}{2m}[\varphi^*_{k'}\nabla\varphi_k - (\nabla\varphi^*_{k'})\varphi_k]\mathrm{e}^{-\mathrm{i}\omega_{kk'}t}$$

- Normally-ordered correlation functions

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1) = \langle \hat{J}^{(-)}(x_1)\cdots\hat{J}^{(-)}(x_n)\hat{J}^{(+)}(y_n)\cdots\hat{J}^{(+)}(y_1)\rangle$$

- $\mathcal{G}^{(n)}(x_1,\ldots,x_n,x_n,\ldots,x_1)$ should be related to registering particles at points $x_1,\ldots,x_n$
- Problem: a field with maximum m particles should yield $\mathcal{G}^{(n)} = 0$ for $n > m$ but does not.

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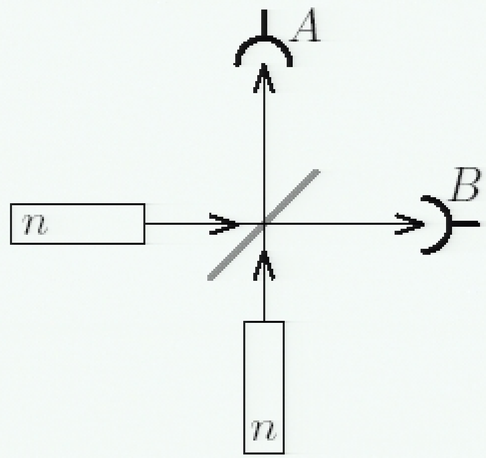

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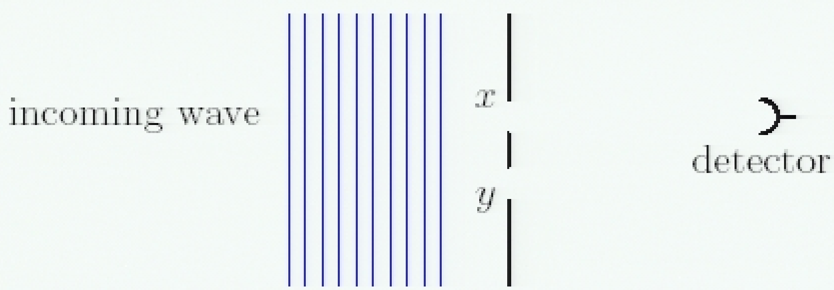
Correlators analogous to bosonic case

- Define correlation functions
$$G^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1)=\langle\hat\psi^\dagger(x_1)\cdots\hat\psi^\dagger(x_n)\hat\psi(y_n)\cdots\hat\psi(y_1)\rangle$$
with $\hat\psi(x_i)$, $\hat\psi^\dagger(x_i)$ fermion annihilation and creation operators at the point x_i, respectively:
$$\hat\psi(\mathbf{r},t)=\sum_k \varphi_k(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega_k t}\hat a_k$$
- $G^{(n>m)}=0$ holds for a field with maximum m particles
- For photons: factorization of $G^{(n)}(x_1,\ldots,x_n,y_n,\ldots,y_1)$ implies coherence
- Magnitude of normalized correlation function
$$g^{(1)}(x,y)=\frac{G^{(1)}(x,y)}{\sqrt{G^{(1)}(x,x)G^{(1)}(y,y)}}$$
expresses visibility of interference fringes in a double-slit experiment:



- Factorization of $G^{(1)}(x,y)$ implies $|g^{(1)}(x,y)|=1$ and hence 100 % visibility

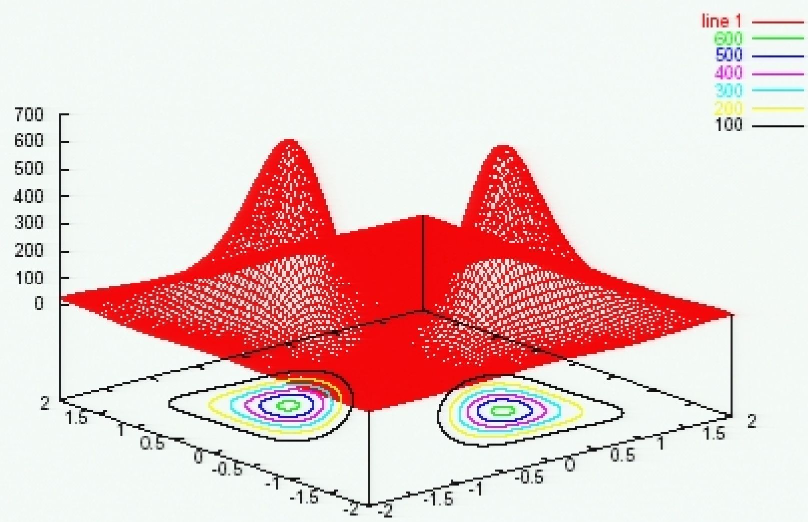
Factorization?

- Factorization of $G^{(n)}$ is impossible for $n > 1$: for any state $|\psi\rangle$

$$\lim_{x_i \to x_j} \langle\psi|\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)\hat{\psi}(y_n)\cdots\hat{\psi}(y_1)|\psi\rangle = 0$$

(for $i \neq j$) – universal antibunching

- Example: multi-mode state with low occupation number per mode



Above: A typical second-order correlation function $G^{(2)}(x, y, y, x)$

- $n > 1$ and $G^{(n)}$ factorizes $\Rightarrow G^{(n)} \equiv 0$
- Perhaps $G^{(1)}(x, y)$ would factorize

Fermionic coherent states

- Displacement operator action on vacuum in analogy with photons

$$D(\gamma)|0\rangle = \mathrm{e}^{\gamma\hat{a}^\dagger - \gamma^*\hat{a}}\,|0\rangle = \cos|\gamma|\,|0\rangle + \sin\gamma \mathrm{e}^{\mathrm{i}\arg(\gamma)}|1\rangle$$

Action of $\hat{D}(\gamma)$ – rotation of the Bloch sphere

- More than one mode:

$$|\vec{\gamma}\rangle_s = \exp\left(\sum_k [\gamma_k \hat{a}_k^\dagger - \gamma_k^* \hat{a}_k]\right)\,|0\rangle$$

This state contains only vacuum and single-particle contributions!

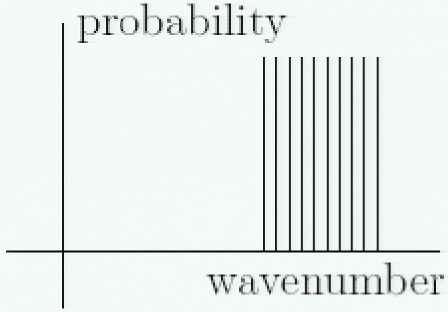
- Consecutive action of displacements instead – problem with non-commuting

$$\mathrm{e}^{\gamma_1\hat{a}_1^\dagger - \gamma_1^*\hat{a}_1}\mathrm{e}^{\gamma_2\hat{a}_2^\dagger - \gamma_2^*\hat{a}_2}\,|0\rangle \neq \mathrm{e}^{\gamma_2\hat{a}_2^\dagger - \gamma_2^*\hat{a}_2}\mathrm{e}^{\gamma_1\hat{a}_1^\dagger - \gamma_1^*\hat{a}_1}\,|0\rangle$$

Physically different states; which is the "right one"?

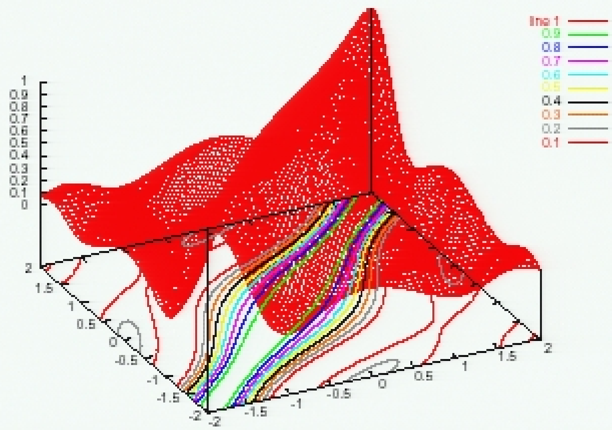
Two examples of fermionic coherent states

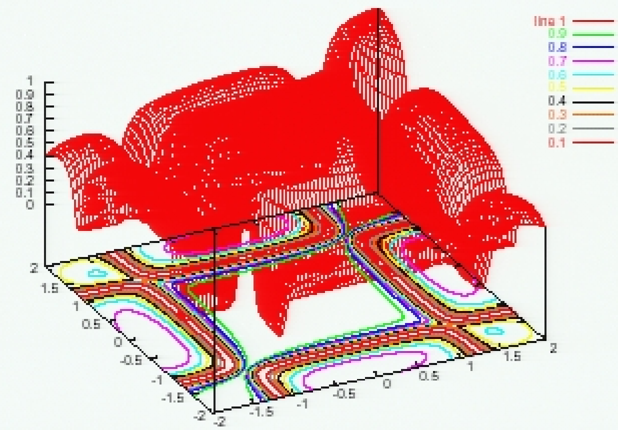
- $N = 40$ modes, average 1.1 particles total ($|\gamma|^2 = 0.0275$ particles per mode)
- Equally-spaced wavenumbers ("comb" spectrum)



$$
\begin{aligned}
|\psi_{\text{ordered}}\rangle &= (\cos\gamma + \sin\gamma \hat{a}_1^\dagger)(\cos\gamma + \sin\gamma \hat{a}_2^\dagger)\cdots(\cos\gamma + \sin\gamma \hat{a}_N^\dagger)|0\rangle \\
|\psi_{\text{random}}\rangle &= (\cos\gamma + \sin\gamma \hat{a}_{i_1}^\dagger)(\cos\gamma + \sin\gamma \hat{a}_{i_2}^\dagger)\cdots(\cos\gamma + \sin\gamma \hat{a}_{i_N}^\dagger)|0\rangle,
\end{aligned}
$$

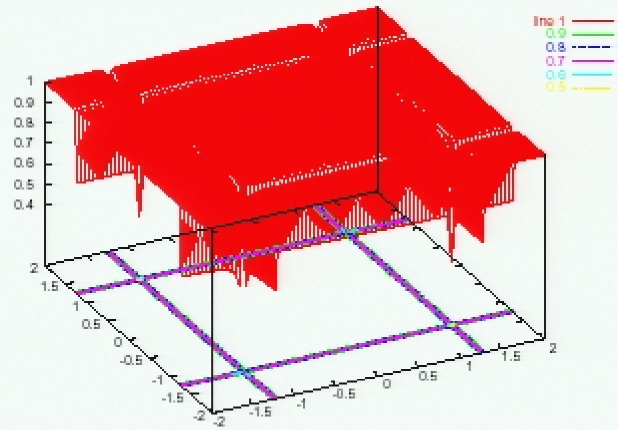
$i_1, \ldots, i_N$ is a permutation of indices $1, 2, \ldots, N$





Above: the normalized first-order correlation function $|g^{(1)}(x,y)|^2$ for the ordered state $|\psi_{\text{ordered}}\rangle$ and randomly ordered state $|\psi_{\text{random}}\rangle$, respectively

- Even for weak states the order matters
- Factorization of $G^{(1)}(x,y)$ occurs in a very small domain only



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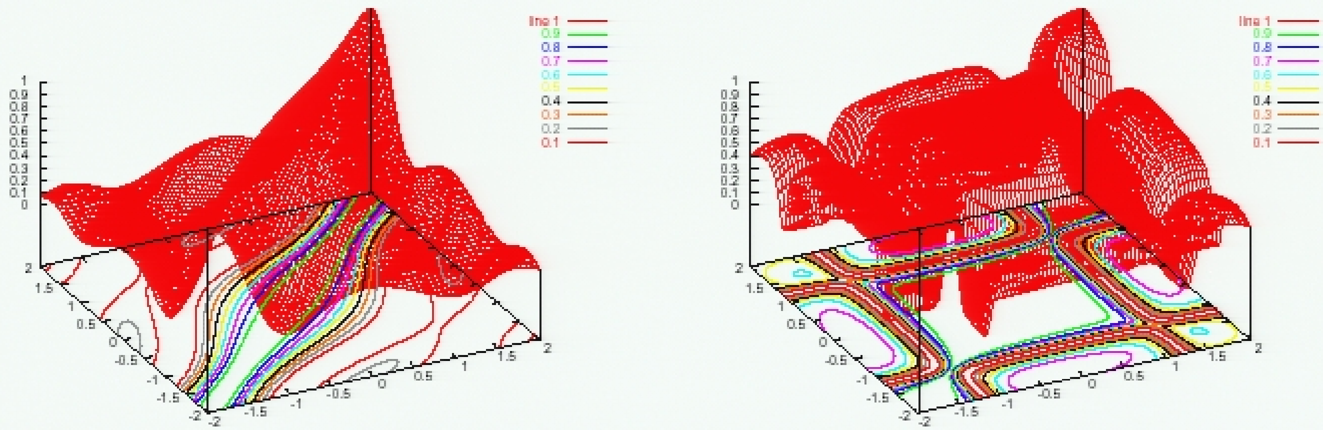

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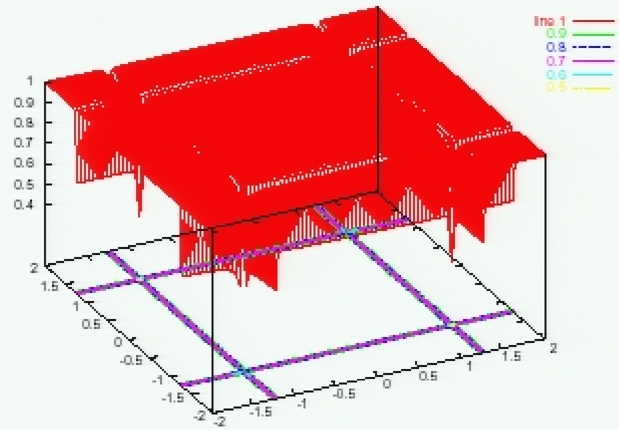

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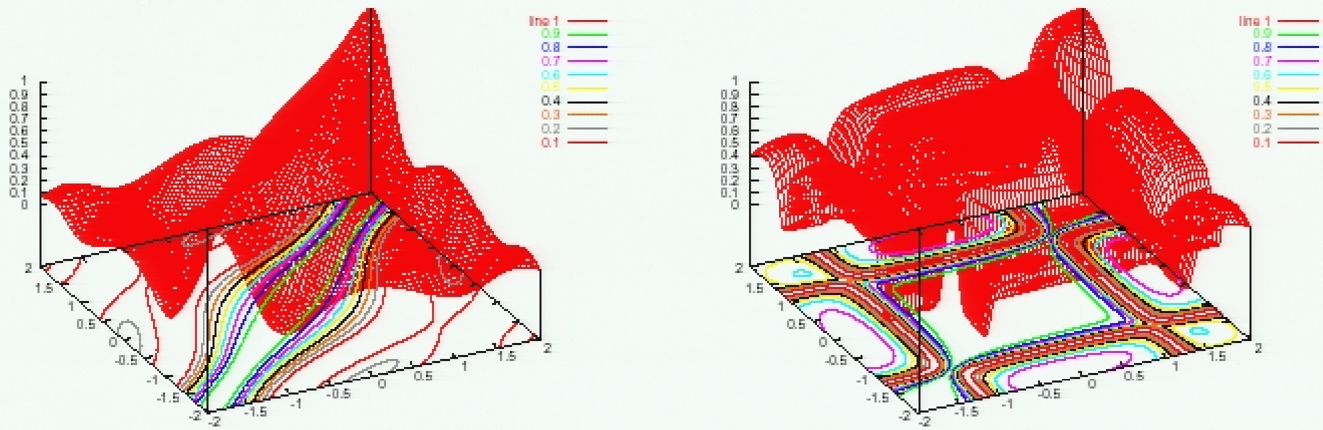

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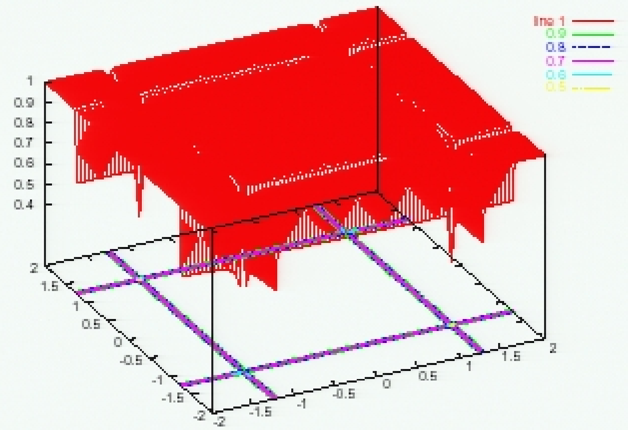

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Problems with fermionic coherent states

- Coherent states we have defined lack the desired properties
- There exist no eigenstates of annihilation operator

$$|\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \quad \Rightarrow \quad \hat{a}|\alpha\rangle = \alpha_1|0\rangle \neq c|\alpha\rangle$$

"Weak states" ($|\alpha_1| \ll 1$) are approximate eigenstates

- No state except vacuum transforms into product state on a beam splitter:

$$(\alpha_0|0\rangle_1 + \alpha_1|1\rangle_1) \otimes |0\rangle_2 \longrightarrow \alpha_0|0\rangle_1|0\rangle_2 + t\alpha_1|1\rangle_1|0\rangle_2 + r\alpha_1|0\rangle_1|1\rangle_2$$

– satisfied by weak states approximately

General fermionic state and sectors

- Problems with fermionic coherent state; most general general pure state is

$$|u\rangle = \left(\alpha + \sum_{i=1}^{n} \alpha_i \hat{a}_i^\dagger + \frac{1}{2} \sum_{i,j=1}^{n} \alpha_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger + \dots \right) |0\rangle$$

$$= \sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k=1}^{n} \alpha_{i_1 i_2 \dots i_k} \hat{a}_{i_1}^\dagger \cdots \hat{a}_{i_k}^\dagger \, |0\rangle.$$

- Superposition of number "sectors" (subspaces of the Hilbert space with exactly n patricles)
- Achieving a general state is a problem, first in creating a superposition of particle number, and also in achieving a superposition in a particular number sector

Coherent states defined in terms of Grassmann variables

- The problems can be avoided by using Grassmann variables
- Grassmann variables mutually anticommute, also anticommute with fermionic creation and annihilation operators, e.g.

$$\xi\eta = -\eta\xi, \qquad \xi^2 = 0, \qquad \xi\hat{a} = -\hat{a}\xi$$

- The vector space of states is larger than the Hilbert space of physical states
- Define a single-mode coherent state as

$$|\xi\rangle = \exp(\hat{a}^\dagger\xi - \xi^*\hat{a})|0\rangle = \exp(\hat{a}^\dagger\xi + \hat{a}\xi^*)|0\rangle = \left(1 - \frac{1}{2}\xi^*\xi\right)|0\rangle - \xi|1\rangle$$

- This state – eigenstate of $\hat{a}$ with eigenvalue ξ:

$$\hat{a}\left[\left(1 - \frac{1}{2}\xi^*\xi\right) - \xi\hat{a}^\dagger\right]|0\rangle \;=\; \xi\hat{a}\hat{a}^\dagger|0\rangle = \xi|0\rangle$$

$$\xi\left[\left(1 - \frac{1}{2}\xi^*\xi\right) - \xi\hat{a}^\dagger\right]|0\rangle \;=\; \left[\xi - \frac{1}{2}\xi\xi^*\xi - \xi^2\hat{a}^\dagger\right]|0\rangle = \xi|0\rangle$$

- Hence also nice transformation on a beam splitter

- Multi-mode coherent state is straightforward:

$$
\begin{aligned}
|\xi\eta\rangle &= \exp(\hat{a}^\dagger\xi - \xi^*\hat{a})\exp(\hat{b}^\dagger\eta - \eta^*\hat{b})|0\rangle \\
&= \exp(\hat{b}^\dagger\eta - \eta^*\hat{b})\exp(\hat{a}^\dagger\xi - \xi^*\hat{a})|0\rangle \\
&= \exp(\hat{a}^\dagger\xi - \xi^*\hat{a} + \hat{b}^\dagger\eta - \eta^*\hat{b})|0\rangle
\end{aligned}
$$

- Order of the exponents does not matter now as $\hat{b}^\dagger\eta$ and $\eta^*\hat{b}$ commute with $\hat{a}^\dagger\xi$ and $\xi^*\hat{a}$
- Very convenient property, the anticommutation is hidden in the Grassmann variables
- Factorization of all the correlation functions is satisfied:

$$
\begin{aligned}
G^{(n)} &= \langle\psi|\hat{\psi}^\dagger(x_1)\cdots\hat{\psi}^\dagger(x_n)\hat{\psi}(y_n)\cdots\hat{\psi}(y_1)|\psi\rangle \\
&= f^*(x_1)\cdots f^*(x_n)f(y_n)\cdots f(y_1)
\end{aligned}
$$

- Agrees with universal antibunching: if $x_i \to x_j$, then $f^*(x_i) \to f^*(x_j)$
- Grassmann coherent states have nice mathematical properties but also some unwanted physical properties (e.g. expectation values of Hermitian operators $\notin \mathbb{R}$)

Conclusion

- Coherence are approached operationally in terms of coherence functions
- Coherent states yield clocks as phase references
- Coherence functions and coherent states are problematic for fermions
- Grassmann variables allow good coherence functions but are not obviously related to physical systems
- Homodyne detection is possible for fermions by splitting and recombining beams – autocorrelation serves as self-referential clock