Title: Entanglement constrained by indistinguishability: a case study om Super-Selection Rules, Reference Frames amd Beyond

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Abstract: Qauntum Information Theory

Entanglement & Ensembleness:
a case study in Super-Selection Rules, Reference Frames and Beyond

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Motivation

We all believe that SSRs and RFs in the context of QI is worth studying. I think entanglement is what makes QI really interesting.

Ensembleness is an interesting constraint on QIP. It can be formulated as a SSR with a finite group, which has not been as studied as SSRs based on Lie groups.

The concept of a RF to break a SSR based on a finite group is not so obvious, so is worth exploring.

Entanglement constrained by this SSR can illustrate analogies with concepts in mixed-state entanglement such as activation by “RF”s.

Finally, there is the question as to whether it is necessary to go beyond SSRs to properly capture ensembleness as in NMR QIP.

Outline

1. What is entanglement?
2. What is a super-selection rule?
3. What is entanglement constrained by a SSR?
4. What is the SSR for ensembleness?
5. How much constrained entanglement is there in ensemble QIP?
6. What is a reference frame for this SSR? (and issues arising)
7. Beyond the SSR: Revisiting the question: How much constrained entanglement is there in ensemble QIP?

1. What is entanglement?

Even considering only *bipartite* entanglement, there are still many concepts of entanglement. One (the strongest?) is

> It is a property of a system shared by two (potentially) distant parties such that their expected measurement correlations are inexplicable without (potentially) faster-than-light signalling.

If entanglement in this sense is present, it will probably also be *useful*, e.g. for dense coding, teleportation, scheduling.

Why such a strong (operational, not mathematical) concept?
To avoid *fluffy bunny entanglement* (Burnett). If entanglement is a mathematical property, then entanglement can be found anywhere just by decomposing Hilbert space in the right way.


I've got a bad feeling about this new entanglement scheme of Bob's

2. What is a SSR?

A SSR is a restriction (fundamental or practical) on the allowed operations $\mathcal{O}$ on a system, **not** on its allowed states.

Note that "operations" includes unitaries, where $\mathcal{O}\rho = \hat{U}\rho\hat{U}^\dagger$, and also measurements, where for example $\mathcal{O}_r\rho = \hat{M}_r\rho\hat{M}_r^\dagger$ and $\sum_r \hat{M}_r^\dagger\hat{M}_r = \hat{1}$.

We define a SSR to be associated with a group G of **local** physical transformations g represented by unitary operators $\hat{T}(g)$.

The G-SSR is the rule that the operations must be G-covariant. That is, they must satisfy

$$\forall\rho \text{ and } \forall g \in G,\ \mathcal{O}[\hat{T}(g)\rho\hat{T}^\dagger(g)] = \hat{T}(g)[\mathcal{O}\rho]\hat{T}^\dagger(g).$$

Relations to traditional SSRs and Conservation Laws

Traditionally (W^3, 1952), one talks of a SSR for an operator (say $\hat{Q}_k$), rather than a SSR associated with a group of transformations.

Such a SSR can be derived from a conservation law for $\hat{Q}$ iff:

1. The state of the universe commutes with global charge $\hat{Q}$.
2. $\hat{Q}$ is of the form $\hat{Q} = \sum_{k=1}^{K} \hat{Q}_k$, where $\hat{Q}_k$ acts on subsystem k.

The resulting SSRs for **local charge** $\hat{Q}_k$ means roughly that it is forbidden to *create* a superposition of states with different Q_k-values.

"SSR for $\hat{Q}_k$" or "$\hat{Q}_k$-SSR" is compatible with our above definition if it is read as "SSR associated with the Lie group G generated by $\hat{Q}_k$."

However, we are also interested in non-Abelian Lie groups such as SU(2), and with *finite* groups such as S_N.

SSRs and Mixing

If a SSR associated with G is in force, then no outcomes will be changed if a state ρ is replaced by the state $\hat{T}(g)\rho\hat{T}^\dagger(g)$ for any $g \in G$.

The most mixed state (that is, the state containing no irrelevant information) with which ρ is physically equivalent is

$$\mathcal{G}\rho \equiv \begin{cases} (\dim G)^{-1}\sum_{g\in G}\hat{T}(g)\rho\hat{T}^\dagger(g), & \text{finite groups} \\ \int_G d\mu_{\text{Haar}}(g)\,\hat{T}(g)\rho\hat{T}^\dagger(g), & \text{Lie groups} \end{cases}.$$

We call this the G-invariant state, as $\forall g \in G, \hat{T}(g)[\mathcal{G}\rho]\hat{T}^\dagger(g) = \mathcal{G}\rho$.

For traditional SSRs, i.e. groups with a single generator $\hat{Q} = \sum_q q\,\hat{\Pi}_q$,

$$\mathcal{G}\rho = \sum_q \hat{\Pi}_q\rho\,\hat{\Pi}_q.$$

3. What is Entanglement Constrained by a SSR?

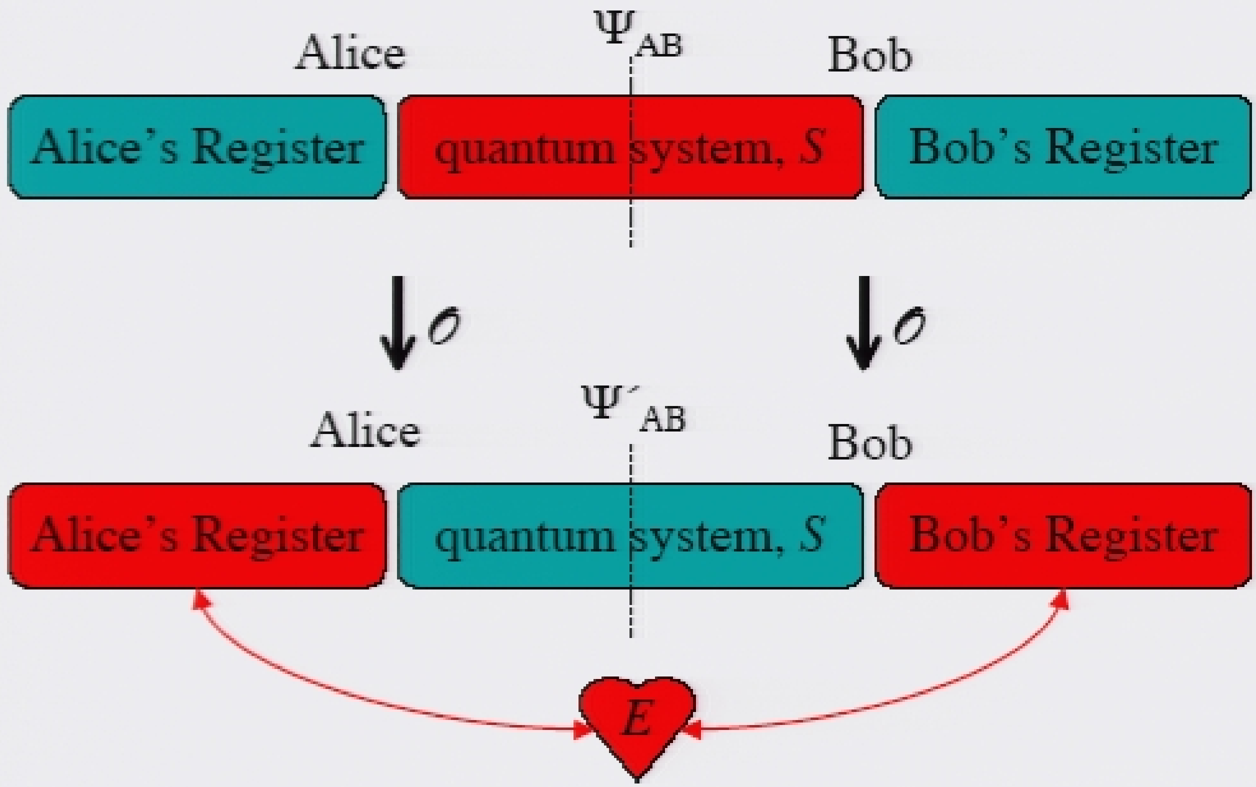
We (Wiseman and Vaccaro, 2003; Bartlett and Wiseman, 2003) proposed an operational definition:

$$E_{G\text{-SSR}}(\rho_{AB}^{\text{sys}}) = \max_{\mathcal{O}} E_{\text{D}}\left(\text{Tr}_{\text{sys}}\left[\mathcal{O}\left(\rho_{AB}^{\text{sys}} \otimes \rho_{AB}^{\text{reg}}\right)\right]\right).$$

- Here we have allowed for a mixed system state ρ_{AB}^{sys}
- The initial register state factorizes: $\rho_{AB}^{\text{reg}} = |\theta_A\rangle\langle\theta_A| \otimes |\theta_B\rangle\langle\theta_B|$.
- The operations $\mathcal{O}$ are G-covariant **local** operations.
- E_{D} for mixed states is the distillable entanglement.

Theorem: The SSR can be enforced by **removing all irrelevant information** from ρ by the decoherence process $\rho \to \mathcal{G}\rho$. That is,

$$E_{G\text{-SSR}}(\rho_{AB}) = E_{\text{D}}\left(\mathcal{G}\rho_{AB}\right).$$



Alice
Ψ_{AB}
Bob
Alice's Register
quantum system, S
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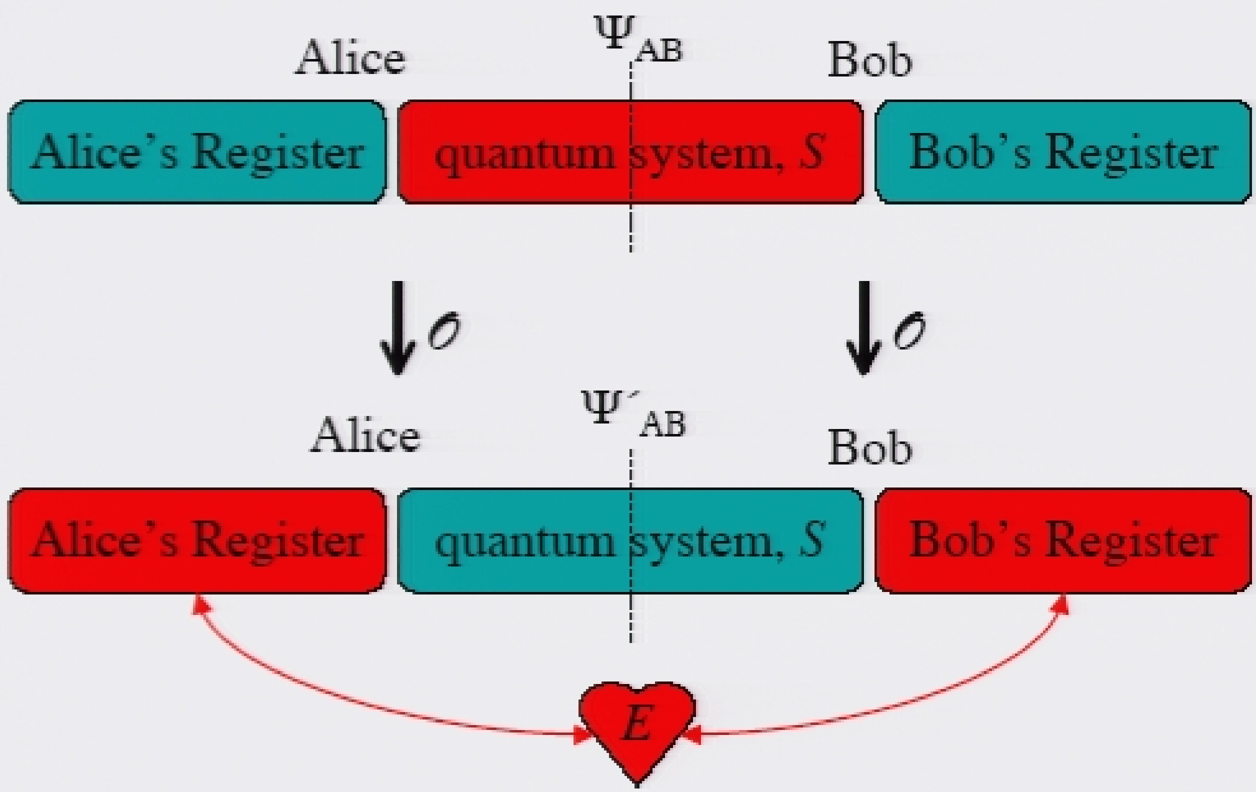

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Proof of Theorem

Denote arbitrary operations by $\mathcal{E}$, and G-covariant operations by $\mathcal{O}$.

$$E_{G\text{-SSR}}(\rho_{AB}^{\text{sys}}) \equiv \max_{\mathcal{O}} E_{\text{D}}\left(\text{Tr}_{\text{sys}}\left[\mathcal{O}\left(\rho_{AB}^{\text{sys}} \otimes \rho_{AB}^{\text{reg}}\right)\right]\right).$$

By G-covariance of $\mathcal{O}$ and properties of trace

$$= \max_{\mathcal{O}} E_{\text{D}}\left(\text{Tr}_{\text{sys}}\left[(\mathcal{G} \circ \mathcal{O} \circ \mathcal{G})\left(\rho_{AB}^{\text{sys}} \otimes \rho_{AB}^{\text{reg}}\right)\right]\right).$$

By G-covariance of $\mathcal{G} \circ \mathcal{E} \circ \mathcal{G}$

$$= \max_{\mathcal{E}} E_{\text{D}}\left(\text{Tr}_{\text{sys}}\left[(\mathcal{G} \circ \mathcal{E} \circ \mathcal{G})\left(\rho_{AB}^{\text{sys}} \otimes \rho_{AB}^{\text{reg}}\right)\right]\right)$$

By properties of trace

$$= \max_{\mathcal{E}} E_{\text{D}}\left(\text{Tr}_{\text{sys}}\left[\mathcal{E}\left(\left[\mathcal{G}\rho_{AB}^{\text{sys}}\right] \otimes \rho_{AB}^{\text{reg}}\right)\right]\right)$$

$$= E_{\text{D}}\left(\mathcal{G}\rho_{AB}^{\text{sys}}\right)$$

4. What is the SSR for ensembleness?

Ensemble QIP (quantum information processing) means:

- N (typically $\gg 1$) identical copies of a "molecule" of M qubits.
- all operations are *symmetric* (i.e. affect each molecule identically).

e.g. NMR: each molecule contains M atoms having a spin-$\frac{1}{2}$ nucleus. For $M = 4$ the qubits could be the spin-$\frac{1}{2}$ nuclei of H, ^{17}O, ^{13}C, ^{19}F. In NMR, the operations use rf magnetic pulses and an antenna.

In NMR QIP it is also the case that the molecules can only be prepared in highly mixed states, and the detection efficiency is very small. However, we regard these limitations as *inessential* to the ensembleness and do not consider them.

The Symmetric Group SSR

The restriction on operations $\mathcal{O}$ can be formulated as the SSR

$$\mathcal{O}[\hat{T}(p)\rho\hat{T}^{\dagger}(p)] = \hat{T}(p)[\mathcal{O}\rho]\hat{T}^{\dagger}(p)$$

Here p is a permutation of the N molecules and $\hat{T}(p)$ is the unitary operator that implements that permutation.

Thus there are M systems (e.g. for $M = 4$, the N H atoms, the N ^{17}O atoms, the N ^{13}C atoms, the N ^{19}F), each of which is acted on $\hat{T}(p)$.

The $N!$ permutations p form the *Symmetric group* S_N.

We define the S_N-invariant (randomly permuted) state

$$\mathcal{P}\rho \equiv \frac{1}{N!}\sum_{p\in S_N} \hat{T}(p)\rho\hat{T}^{\dagger}(p).$$

A Simple Example

Say $M = 3$ (nuclei A, A' and B, per molecule) and $N = 2$ (there are two molecules, 1 and 2), and the state is $|\psi\rangle = |\uparrow^1_A \uparrow^1_{A'} \uparrow^1_B\rangle |\downarrow^2_A \downarrow^2_{A'} \downarrow^2_B\rangle$.

We consider that the As and A's belong to Alice and the Bs to Bob, and the S_2-SSR applies independently to Alice and to Bob. Now if Alice's local operations (acting only on As and A's) cannot distinguish molecules 1 and 2, then this state is equivalent to

$$\hat{T}_A(1 \leftrightarrow 2)|\psi\rangle = |\downarrow^1_A \downarrow^1_{A'} \uparrow^1_B\rangle |\uparrow^2_A \uparrow^2_{A'} \downarrow^2_B\rangle.$$

Under the action of $\mathcal{P}_A$ (or $\mathcal{P}_B$), $|\psi\rangle$ goes to an equal mixture:

$$|\psi\rangle \xrightarrow{\mathcal{P}_A \otimes \mathcal{P}_B} \frac{1}{\sqrt{2}} |\uparrow^1_A \uparrow^1_{A'} \uparrow^1_B\rangle |\downarrow^2_A \downarrow^2_{A'} \downarrow^2_B\rangle \uplus \frac{1}{\sqrt{2}} |\downarrow^1_A \downarrow^1_{A'} \uparrow^1_B\rangle |\uparrow^2_A \uparrow^2_{A'} \downarrow^2_B\rangle.$$

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More General Action of $\mathcal{P}$

Consider the case of $N = 2J$ spin-$\frac{1}{2}$ particles (i.e. just one party and $M = 1$). Then the total Hilbert space can be decomposed as

$$\mathbb{H}_2^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathbb{H}_{jR} \otimes \mathbb{H}_{jP}.$$

- $\text{Dim}(\mathbb{H}_{jR}) = 2j+1$ (where j is the "total angular momentum"), and joint operations such as rotations act only upon $\mathbb{H}_{jR}$.
- $\text{Dim}(\mathbb{H}_{jP}) = d_j \equiv \binom{N}{N/2-j}\frac{2j+1}{N/2+j+1}$, and permutations of the spins $\hat{T}(p)$ act only upon $\mathbb{H}_{jP}$.

This gives the following basis for $\mathbb{H}_2^{\otimes N}$: $\left\{|j,m\rangle \otimes |j,p\rangle : {}_{j=0}^{J}; {}_{m=-j}^{j}; {}_{p=1}^{d_j}\right\}$. The action of $\mathcal{P}$ is to mix over the states $|j,p\rangle$.

Action of $\mathcal{P}$ on an Entangled State

This was actually done first by Eisert *et al.* (PRL, 2000). Each molecule consists of two nuclei, and all are prepared identically:

$$
\begin{aligned}
|\psi\rangle &= (\alpha|\downarrow_A\downarrow_B\rangle + \beta|\uparrow_A\uparrow_B\rangle)^{\otimes N} \\
&= \sum_{j=0}^{J}\sum_{m=-j}^{j}\sum_{p=1}^{d_j} \alpha^{J-m}\beta^{J+m}|j,m\rangle_A|j,p\rangle_A \otimes |j,m\rangle_B|j,p\rangle_B.
\end{aligned}
$$

In this case $E(|\psi\rangle) = N(-\alpha^2\log\alpha^2 - \beta^2\log\beta^2)$. But under S_N-SSR,

$$
|\psi\rangle \xrightarrow{\mathcal{P}_A\otimes\mathcal{P}_B} \biguplus_{j=0}^{J} \biguplus_{p_A,p_B=1}^{d_j} \sqrt{\wp_j} \sum_{m=-j}^{j} \alpha^{j-m}\beta^{j+m}|j,m\rangle_A|j,p_A\rangle_A \otimes |j,m\rangle_B|j,p_B\rangle_B,
$$

where $\wp_j = \sum_{m=-j}^{j}\alpha^{2(J-m)}\beta^{2(J+m)}/d_j$.

5. How much constrained entanglement is there in ensemble QIP?

$$E_{S_N\text{-SSR}}(|\psi\rangle) = E_D(\mathcal{P}_A \otimes \mathcal{P}_B[|\psi\rangle\langle\psi|]) = \sum_{j=0}^{J} d_j^2 \wp_j E(|\phi_j\rangle),$$

where $|\phi_j\rangle = \sum_{m=-j}^{j} \alpha^{j-m}\beta^{j+m} |j,m\rangle_A \otimes |j,m\rangle_B$ [Eisert *et al.*, 2000].

Consider the particular case of Bell states, where $\alpha = \beta = \frac{1}{\sqrt{2}}$. Then $E(|\psi\rangle) = N$, but, as shown by Bartlett and Wiseman (2003),

$$E_{S_N\text{-SSR}}(|\psi\rangle) = \sum_{j=0}^{J} \frac{(2j+1)}{2^{2J}} \binom{2J}{J-j} \frac{2j+1}{J+j+1} \log_2(2j+1) \sim \frac{1}{2}\log_2 N.$$

Since this is the maximum entanglement, the entanglement per molecule must always $\to 0$ as $N \to \infty$.

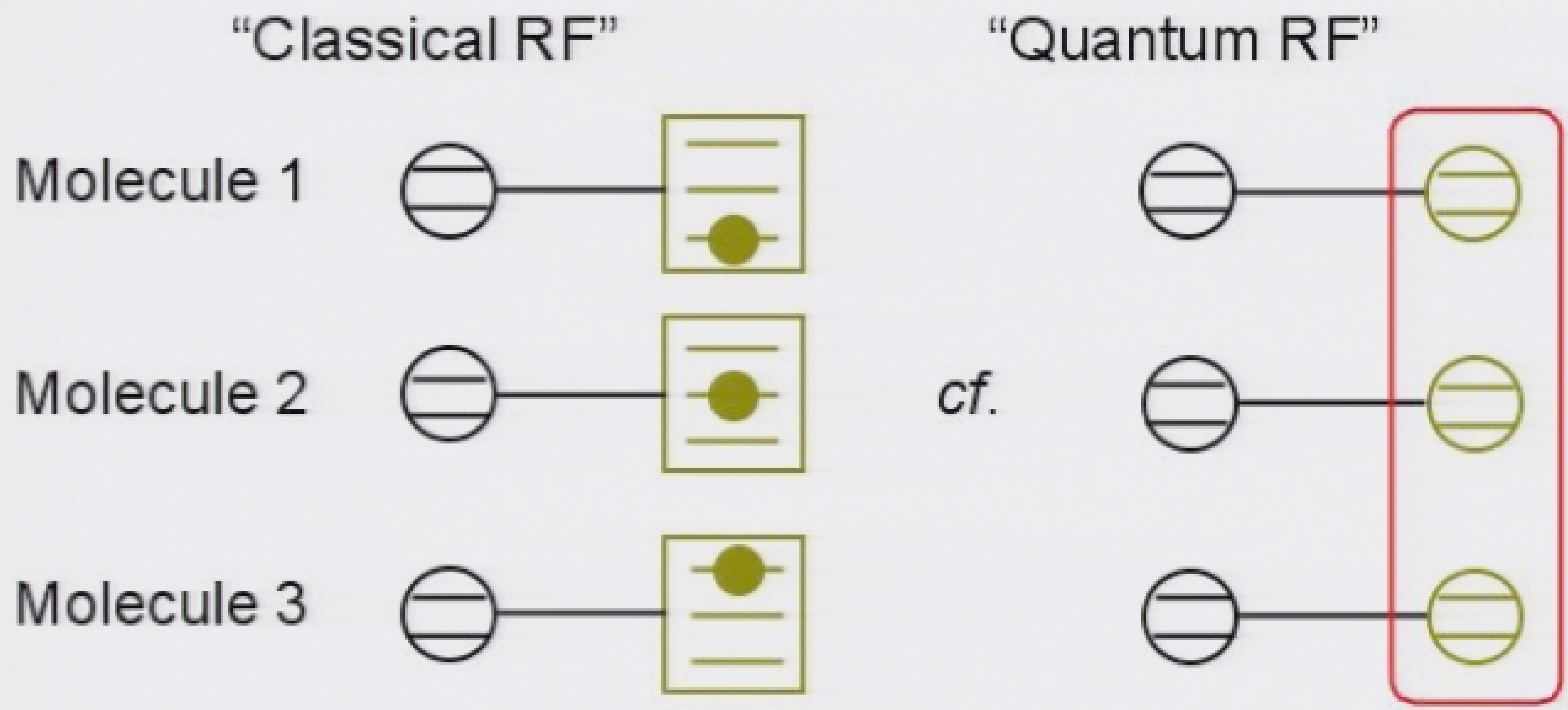
6. What is a reference frame for this SSR?

In general, a RF for a SSR is something that ameliorates (at least in part) its effect.

Because S_N is a *finite* group, a perfect RF can be finite. The simplest example is just a way to label each molecule, such as by an extra nucleus (or group of nuclei) with many levels. For example, with $N = 3$, a state with reference frame is

$$|\Psi\rangle = |\psi^1, 1\rangle|\psi^2, 2\rangle|\psi^3, 3\rangle = |\psi^1, \psi^2, \psi^3\rangle \otimes |1, 2, 3\rangle$$

Here $|\psi^k\rangle$ is the state of the M nuclei in the kth molecule (not including the RF) which we have assumed to factorize.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Quantum RFs?

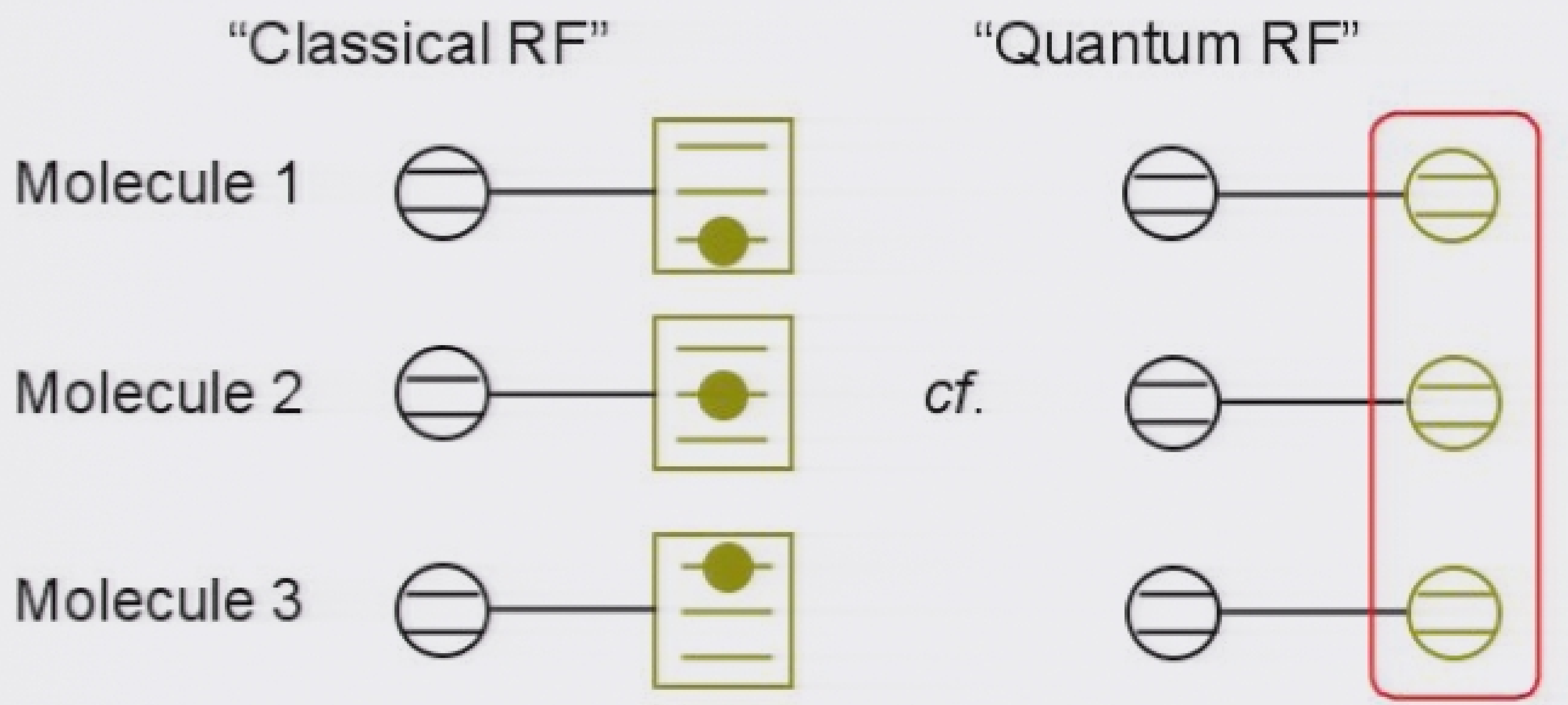
The Hilbert space dimension of the RF just introduced is N^N.

cf. the dimension of S_N which is $N! \approx (N/e)^N$ (Stirling).

This suggests that it might be possible to use a *smaller* RF if we allow for *entanglement* between the different molecules.

It turns out that this is indeed the case [Korff and Kempe, quant-ph/0405086]. In the limit $N \to \infty$, a RF of size d^N with $d = \lfloor Nr \rfloor$ works perfectly for any $r > 1/e$.

Also, for $N = 3$ they give an example of a 2^3-dimension RF which works with probability $5/6$ (compared to $1/2$ classically).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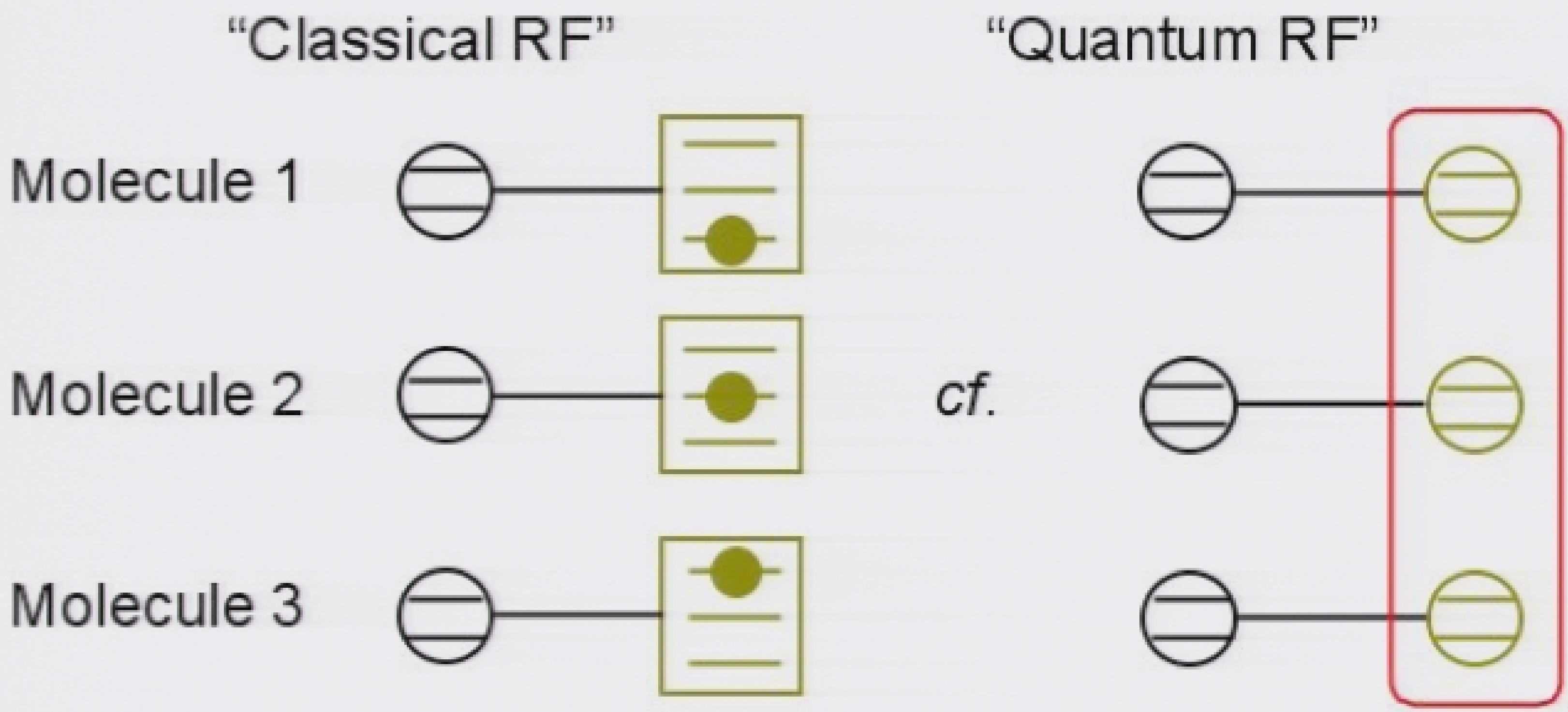
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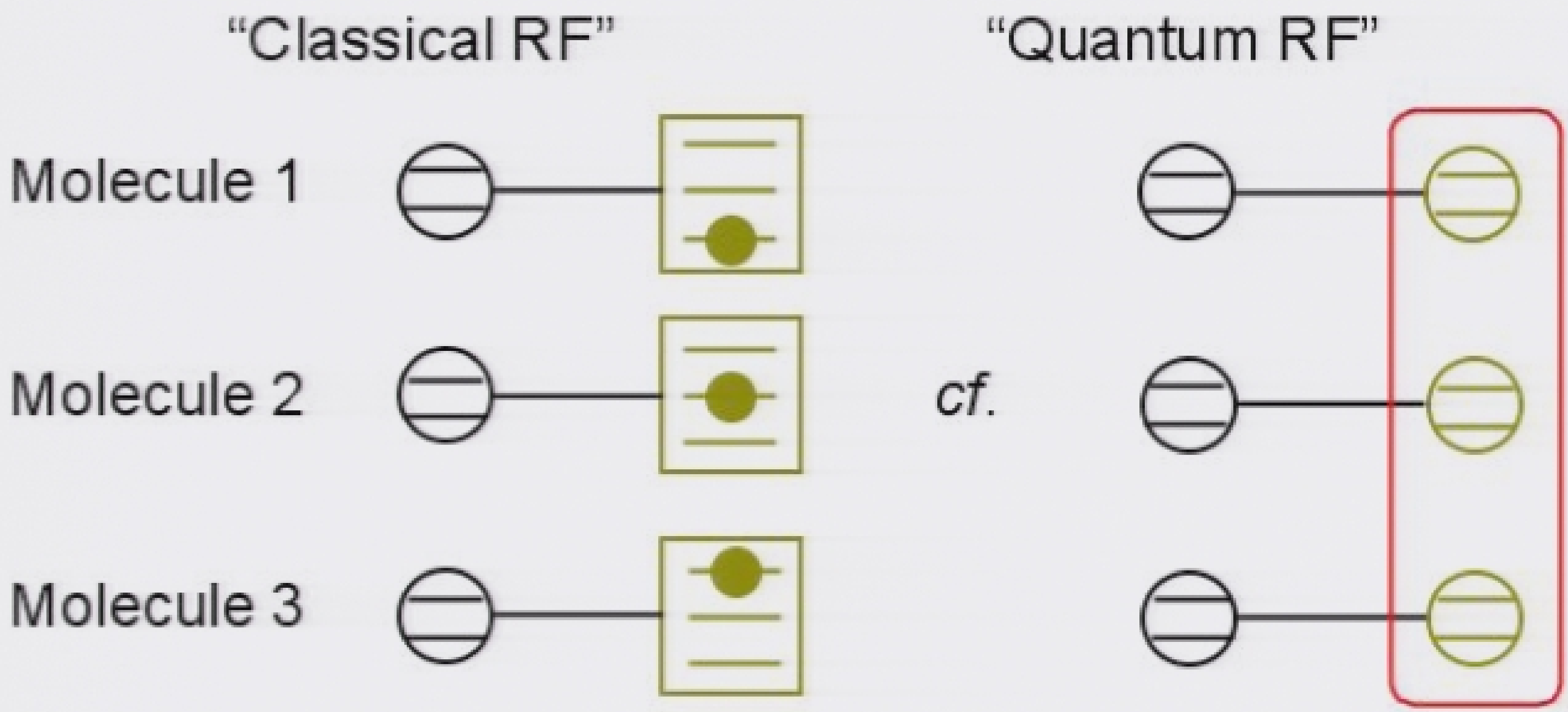
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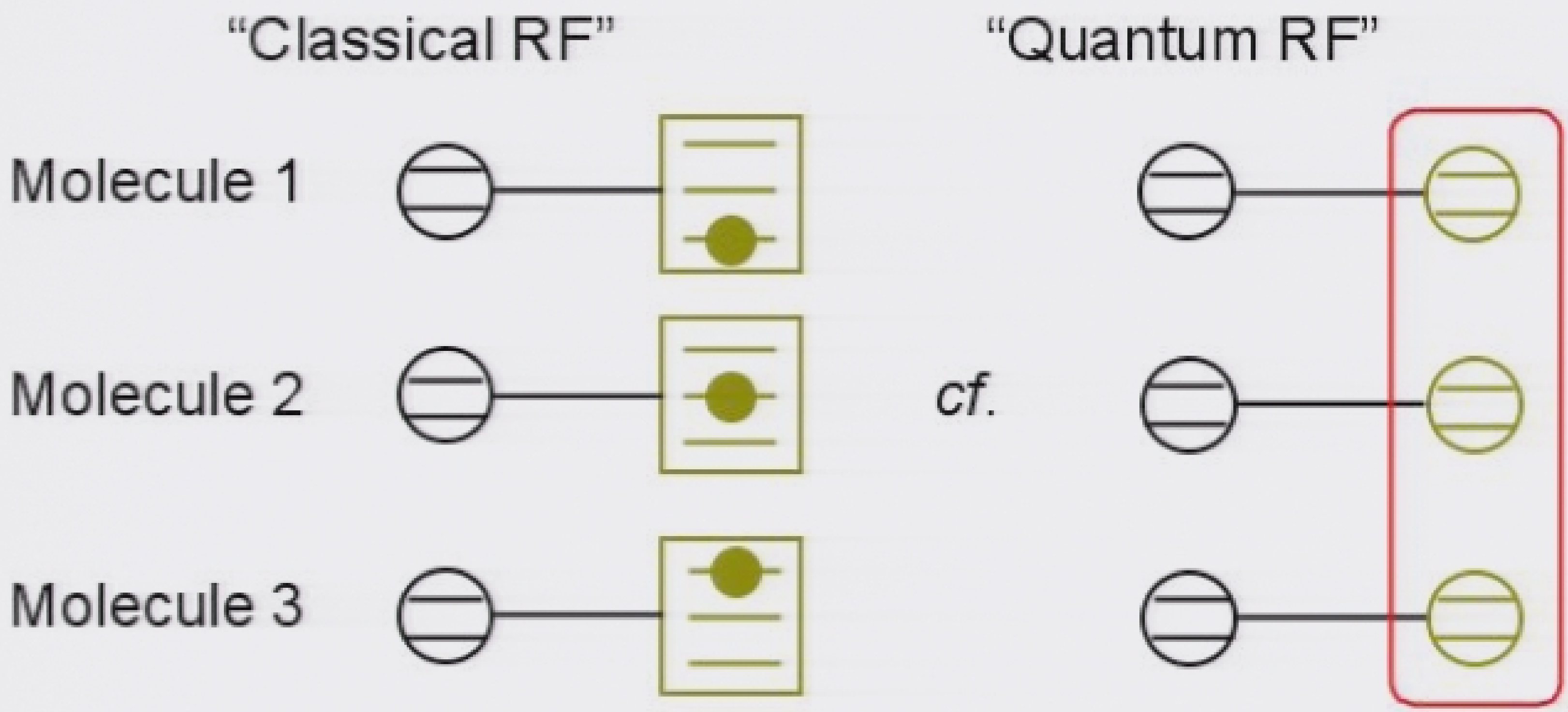
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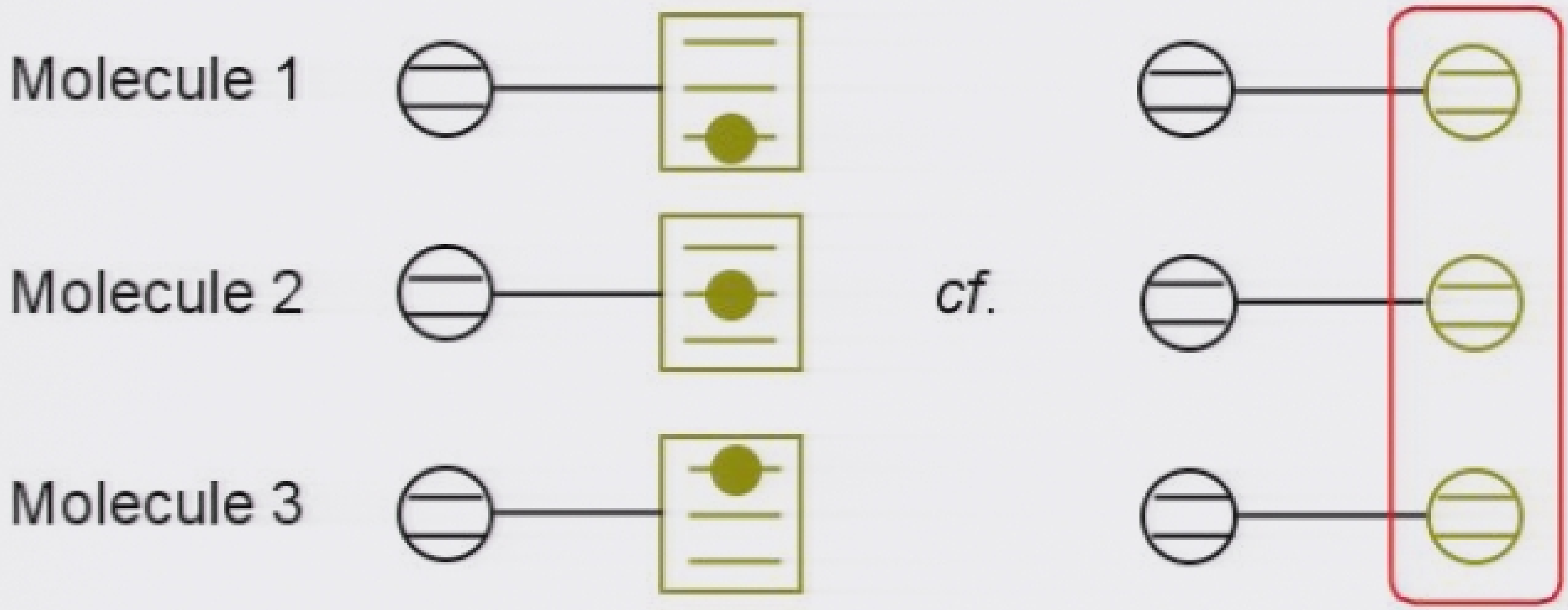

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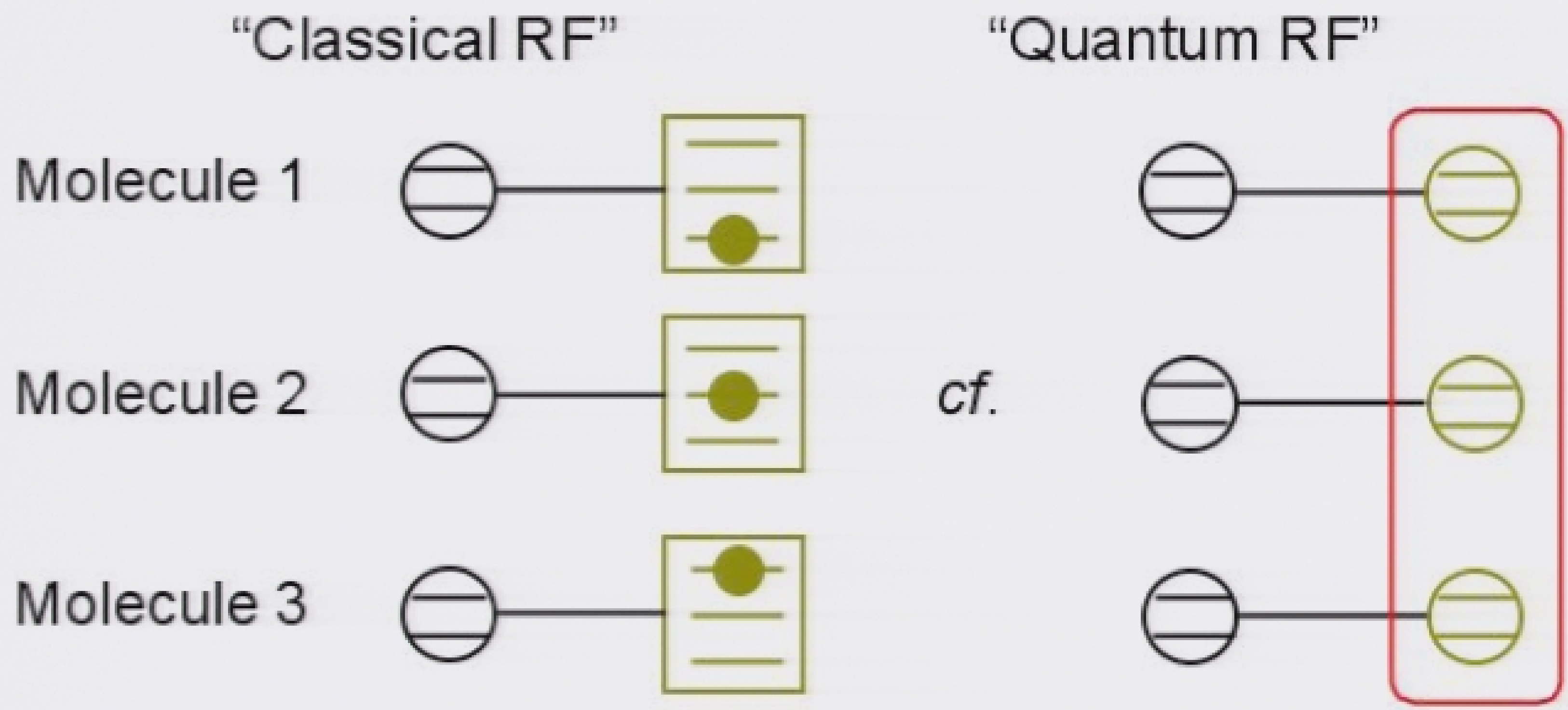
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Shared RFs

The simplest shared RF is for Alice and Bob each to have a RF:[1]

$$\begin{aligned} |\Psi\rangle &= |\psi^1_{AB}, 1_A, 1_B\rangle|\psi^2_{AB}, 2_A, 2_B\rangle|\psi^3_{AB}, 3_A, 3_B\rangle \\ &= |\psi^1_{AB}, \psi^2_{AB}, \psi^3_{AB}\rangle \otimes |1_A, 1_B, 2_A, 2_B, 3_A, 3_B\rangle \end{aligned}$$

Although these states are separable, they cannot be prepared *locally* by $\mathcal{P}_A \otimes \mathcal{P}_B$-covariant operations from a $\mathcal{P}_A \otimes \mathcal{P}_B$-invariant state.

Therefore, shared RFs in the presence of the S_N-SSR are a form nonlocal resource, that can lift the S_N restriction of entanglement. [*cf.* Verstraete and Cirac (2003) for $U(1)$]

[1]Note that such states are not globally $\mathcal{P}$-invariant. However, using the final RF basis above we can write a $\mathcal{P}$-invariant RF: $\biguplus_{k=1}^{3} \frac{1}{\sqrt{3}}|k\rangle_A|k\rangle_B|k+1\rangle_A|k+1\rangle_B|k+2\rangle_A|k+2\rangle_B$, where "+" means "+ mod 3".

Entanglement Analogies: Activation

These non-(locally preparable) but separable states are analogous to nonseparable PPT states in the theory of mixed-state entanglement.

They can *activate* the entanglement of nonseparable pure states with $E_{S_N\text{-SSR}} = 0$, which are analogous to non-1-distillable NPPT states.

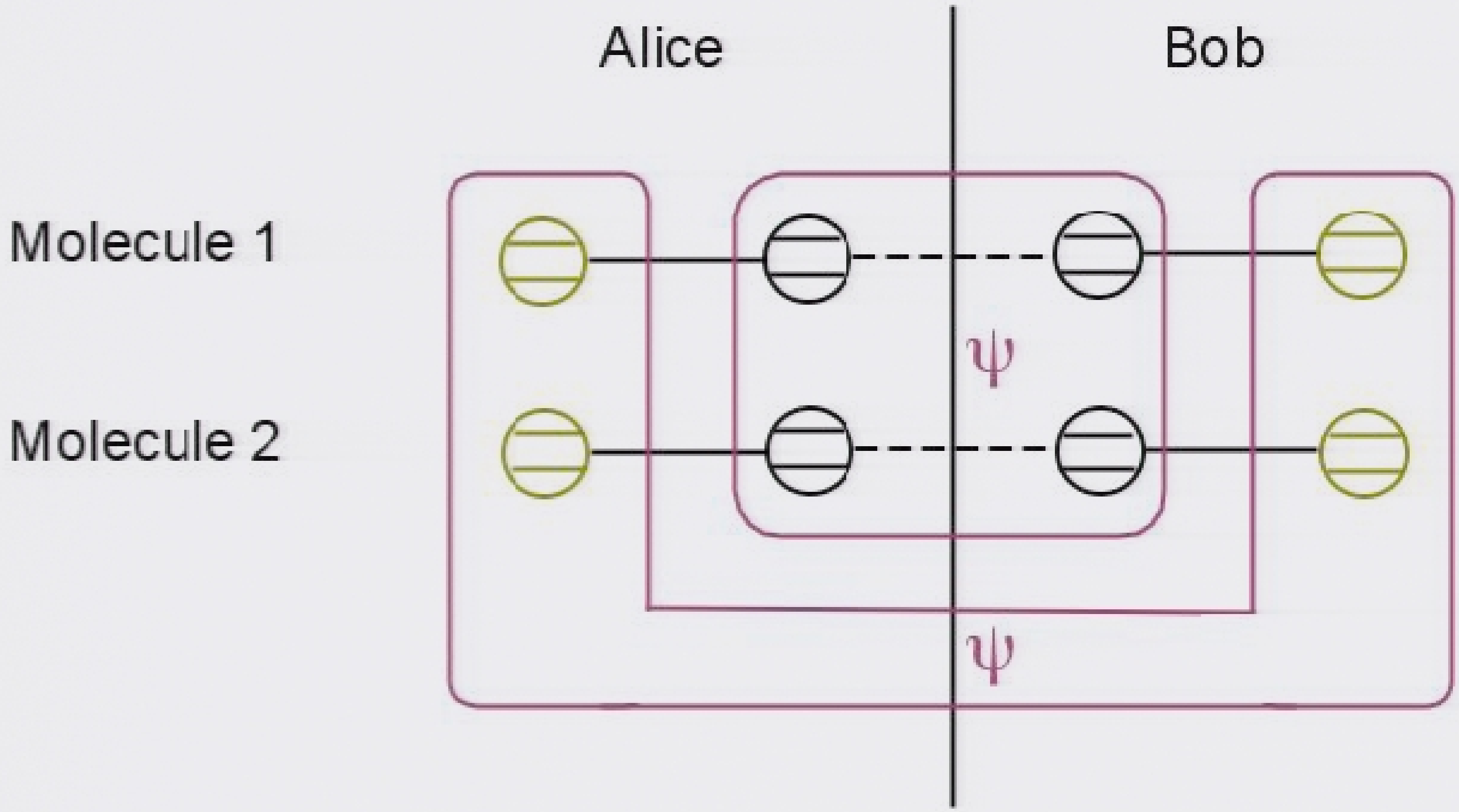
An example of such a state is the following. With $N = 2$ and $M = 2$, with Alice and Bob owning one nucleus per molecule,

$$\sqrt{2}|\psi\rangle = |+\rangle_A|-\rangle_B + |-\rangle_A|+\rangle_B$$

Here $|+\rangle = |j = 1, m = 0\rangle$ and $|-\rangle = |j = 0, m = 0\rangle$, so $\hat{T}(1 \leftrightarrow 2)|\pm\rangle = \pm|\pm\rangle$, and

$$\sqrt{2}|\psi\rangle \xrightarrow{\mathcal{P}_A \otimes \mathcal{P}_B} |+\rangle_A|-\rangle_B \uplus |-\rangle_A|+\rangle_B.$$

One copy of a state acts as a quantum RF for another copy



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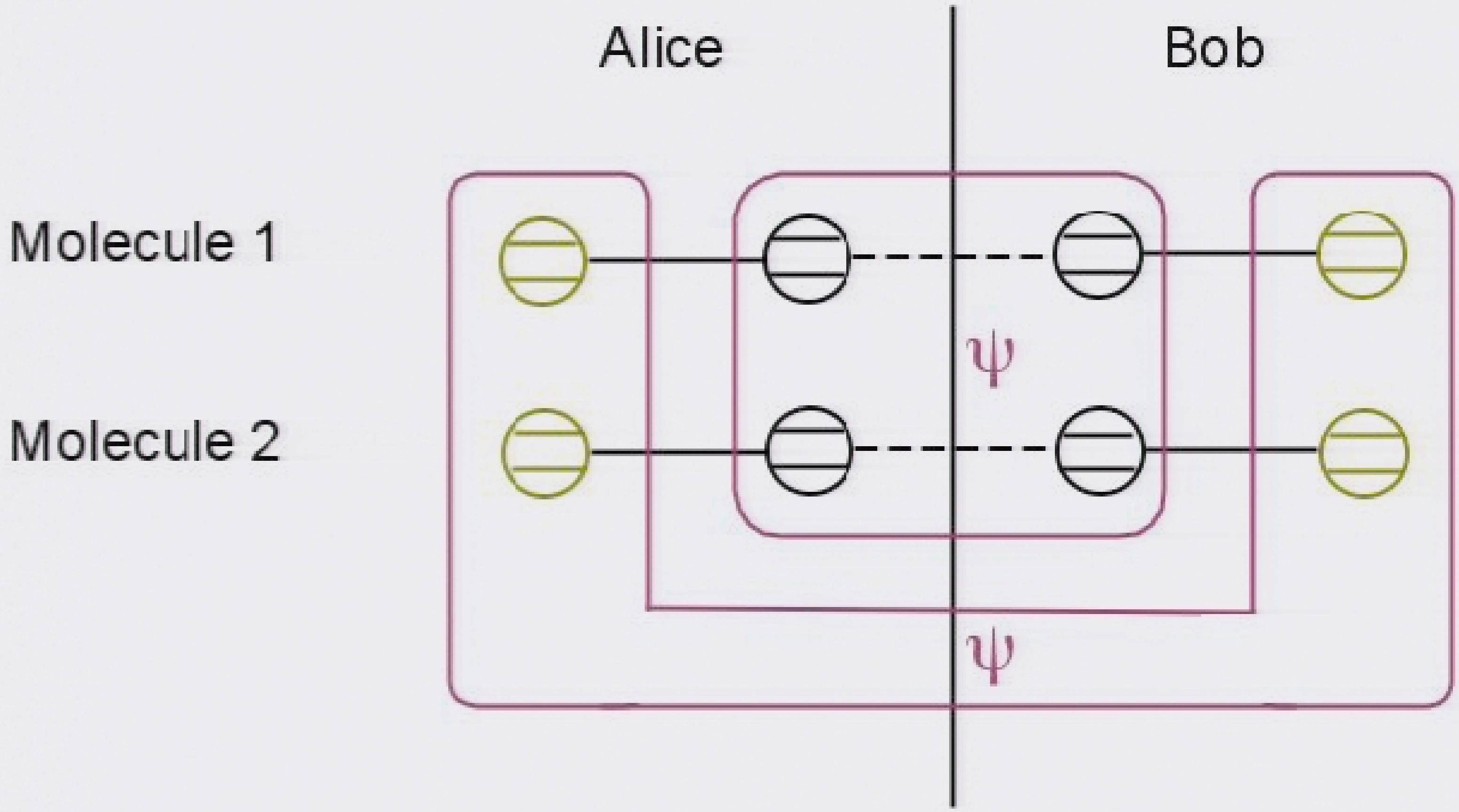

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Entanglement Analogies: 1-D $\not\equiv$ 2-D

Although $\sqrt{2}|\psi\rangle = |+\rangle_A|-\rangle_B + |-\rangle_A|+\rangle_B$ has $E_{S_2\text{-SSR}} = 0$, with two copies some entanglement can be obtained.

Note that two copies does *not* mean four molecules. Since S_2 is fixed, we still have $N = 2$ molecules. But instead of $M = 2$, now $M = 4$, with Alice and Bob each having *two* nuclei:

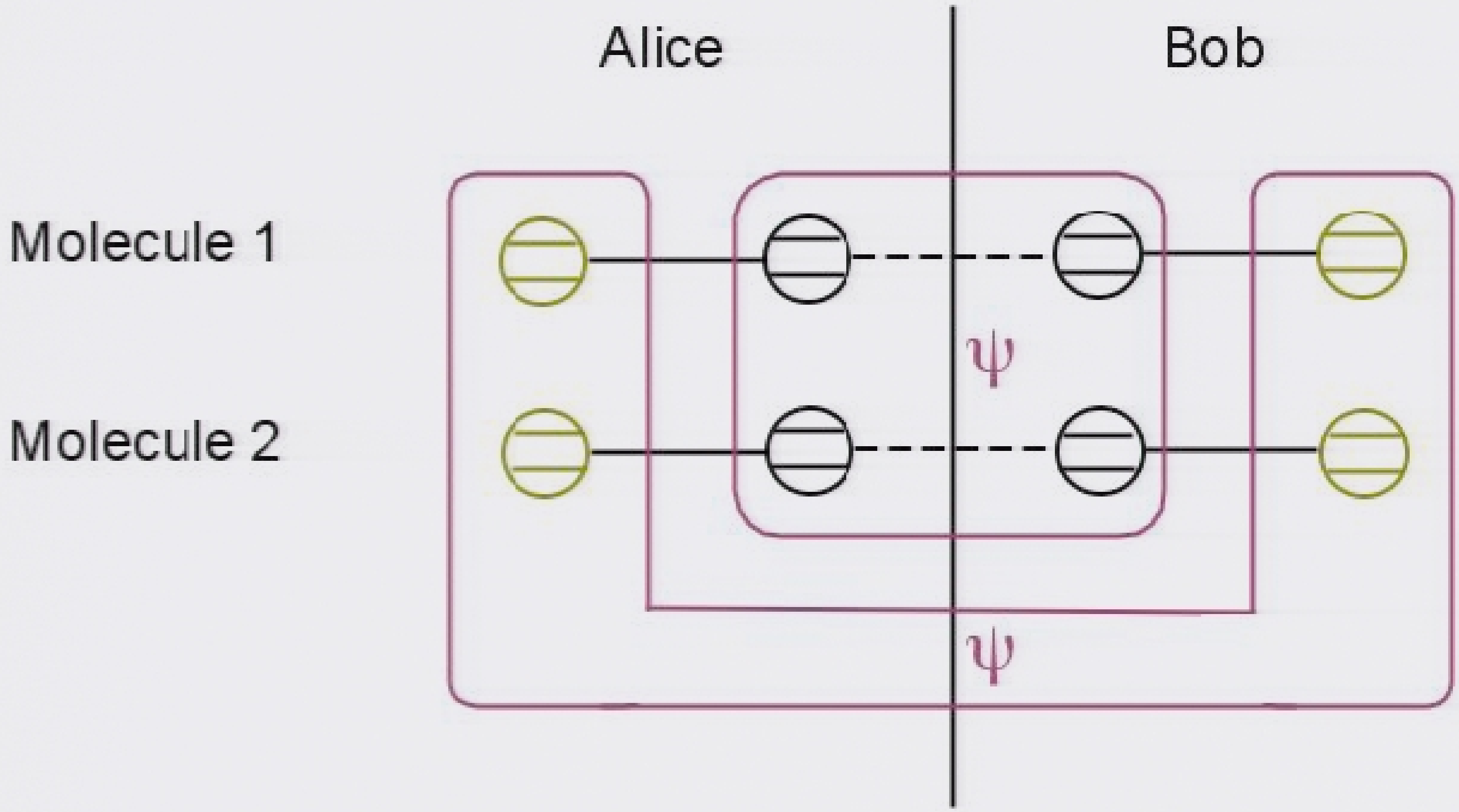
$$(\sqrt{2}|\psi\rangle)^{\otimes 2} = |++\rangle_A|--\rangle_B + |-+\rangle_A|+-\rangle_B + |+-\rangle_A|-+\rangle_B + |--\rangle_A|++\rangle_B.$$

The constrained entanglement of this state is 1 ebit:

$$(\sqrt{2}|\psi\rangle)^{\otimes 2} \xrightarrow{\mathcal{P}_A\otimes\mathcal{P}_B} |++\rangle_A|--\rangle_B + |--\rangle_A|++\rangle_B \uplus |-+\rangle_A|+-\rangle_B + |+-\rangle_A|-+\rangle_B.$$

This is analogous to the existence of distillable NPPT states that are not 1-distillable. That is, non-1-D NPPT states can act as RFs too.

One copy of a state acts as a quantum RF for another copy



7. Beyond the S_N-SSR

The S_N-SSR says that all elements (molecules) are subject to identical operations. But is this enough to characterize NMR QIP?

Totally symmetric operations also characterize spin-squeezing experiments, but in these experiments, *collective* operations on the elements (atoms) are also possible. For example, collective QND measurement of the atomic spins entangles the atoms.

I suggest NMR QIP is more constrained in that operations must be non-collective as well as symmetric. Considering the $M = 1$ for simplicity, then all that can be done in practice is

- Rotations $\exp(-i\underline{\theta} \cdot \underline{\hat{J}}) = \exp(-i\underline{\theta} \cdot \sum_{k=1}^{N} \underline{\hat{\sigma}}^k/2)$.
- Destructive measurement of $\hat{J}_z = \sum_{k=1}^{N} \hat{\sigma}_z^k/2$.

How can we characterize these constraints?

I don't know the definitive answer but
(1) I suspect it is *not* by any G-SSR
(2) A stronger (but not too strong) constraint can be found:

- If the molecules are **randomly permuted locally** (i.e. the state is operated upon by $\mathcal{P}_A \otimes \mathcal{P}_B$) then the entanglement obtained cannot be more than under the S_N-SSR.
- NMR operations *could* be performed by thus permuting **and then separating** the molecules and performing unitaries and measurements on them independently.
- So, $\mathcal{P}_A \otimes \mathcal{P}_B$ followed by separate operations is a stronger constraint than the S_N-SSR, but is not too strong for NMR QIP.

Revisiting the question: How much constrained entanglement is there in ensemble QIP?

Consider the case (as in NMR QIP) where all molecules are prepared in the same state, ρ_{AB}.

If they are randomly permuted and then separated, then there are now N “Alices” and N “Bobs”, and any pair consisting of one Alice and one Bob shares the state

$$\rho'_{AB} = \frac{1}{N}\rho_{AB} + \frac{N-1}{N}\mathrm{Tr}_B[\rho_{AB}] \otimes \mathrm{Tr}_A[\rho_{AB}].$$

The total entanglement shared between the Alices and the Bobs is thus bounded above by $E'_N = N \times E_{\mathrm{F}}(\rho'_{AB})$, and this is also an upper bound on the “true” entanglement constrained by ensembleness.

Progress so far

Say $M = 2$ and each molecule is prepared as $\alpha|\downarrow_A\downarrow_B\rangle + \beta|\uparrow_A\uparrow_B\rangle$. Then

$$\begin{aligned}\rho'_{AB} \;=\; & \frac{1}{N}(\alpha|\downarrow_A\downarrow_B\rangle + \beta|\uparrow_A\uparrow_B\rangle)(\alpha\langle\downarrow_A\downarrow_B| + \beta\langle\uparrow_A\uparrow_B|) \\ & + \frac{N-1}{N}(\alpha^2|\downarrow_A\rangle\langle\downarrow_A| + \beta^2|\uparrow_A\rangle\langle\uparrow_A|)\otimes(\alpha^2|\downarrow_B\rangle\langle\downarrow_B| + \beta^2|\uparrow_B\rangle\langle\uparrow_B|).\end{aligned}$$

We (Jones and Wiseman) do not have an analytical expression for $E'_N = N \times E_{\mathrm{F}}(\rho'_{AB})$ but we do know the following:

- For $\alpha = \beta = \frac{1}{\sqrt{2}}$, $E'_1 = 1$, $E'_2 = 0.235\cdots$, and for $N > 2$, $E'_N = 0$.
- For $N \gg 1$ it seems the maximum $E'_N = O(\log(N)/N^3)$ for $\alpha = O(1/N)$

CONCLUSIONS

Entanglement and Ensembleness highlights some interesting issues:

- How should constrained entanglement be defined?
- What is a reference frame, and is there a difference between quantum and classical RFs?
- Can constrained entanglement and RFs be related to unconstrained entanglement for mixed states?
- Is it necessary to go beyond SSRs to capture interesting constraints?

Future Work

- Obtain analytical results for E'_N, at least asymptotically.
- Go "beyond beyond", by considering what one can really do in ensemble QIP to use (or at least demonstrate) entanglement.
- Think about whether this simple non-SSR constraint is at all useful in thinking about other non-SSR constraints such as LOCC.
- Link in with other work with John Vaccaro, Steve Bartlett and Rob Spekkens, ...?