Title: Generalized entanglement and superselection rules

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Abstract: Qunatum Information Theory

A unifying generalization: entanglement relative to subspaces of observables

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Our program

- Generalize notions like entanglement, "local operations and classical communication" (class of physical operations that do not increase entanglement), measures of entanglement, asymptotic measures of entanglement, etc. to our setting.
- Use GE to understand issues in condensed matter, e.g. quantum phase transitions, applicability of mean-field-like theories based on quasiparticles. Do the quasiparticles interact sufficiently weakly? How much "generalized entanglement" of the quasiparticles exists in the ground state?
- Other special structures of observable

Outline

1. Generalize entanglement: generalized entangled pure states are those that are mixed with respect to a reduced set of observables.

2. A measure: quadratic purity of reduced state

3. Examples: Spin-S representations of $\mathfrak{su}(2)$; NMR fermionic $U(N)$, $SO(2N)$.

4. General semisimple Lie algebras

5. Measures of generalized entanglement

6. Generalizing notions of local (unilocal, separable, locc) maps

7. Application in anisotropic XY model

Our generalization

- Setting: Quantum system with *distinguished subspace of the full space of "observables" (Hermitian operators).* Generalized entanglement is **relative** to a choice of reduced observable space. Example: To recover standard bipartite entanglement, "Local" observables: linear combinations of $X^A \otimes I$, $I \otimes Y^B$, but *not* $X^A \otimes Y^B$.
- A state gives a set of expectation values of a basis for the traceless observables. (E.g. Bloch representation of spin-1/2 density matrix in terms of expectation values of $\sigma_x, \sigma_y, \sigma_z$.)
- Any state induces a set of expectation values of the **reduced** set of observables: a "reduced

- The set of states, and the set of reduced states, form "nice" (closed, pointed) convex positive cones. The normalized ones form a convex set, so there is a notion of pure (extremal) ones.

  [Note: the reduced states in general form a smaller cone than the full cone of positive linear functionals on the reduced observable space]

  Beyond QM: abstract cones

- **Definition of generalized entangled pure state:** one that induces a **mixed** reduced state.

- Extend to mixed states as in standard case.

- Motivation: reduced observable space may represent physically natural restricted means

Important special case: reduced observable space is a Lie algebra $\mathfrak{g}_o$ of traceless Hermitian operators. "Distinguished family of Hamiltonians."

- Lie group $e^{\mathfrak{g}}$: Interpret skew-Hermitian part as generating unitary dynamics via $e^{\mathfrak{g}_o}$, other exponentials (and limits of them!) as Lie-algebraically definable Kraus operators.

The purity

- One (inverse) measure of generalized entanglement: "purity $\mathrm{tr}\rho^2$ of *reduced state* ρ. Can be defined for reduced states in Lie algebra setting, theorem below states that it's maximal for pure (extremal) states of the reduced algebra. (irreps)

- Sum of squared expectation values of operators in a basis for our algebra or subspace. Invariant under *unitary* change of basis in our operator subspace.

- Lie-algebraic case: e.g., sum of squared expectation values of a Hermitian orthonormal (in the trace, i.e. Killing form, inner product $\mathrm{tr}AB$) basis for the Lie algebra. Invariant under the

3-d representation of su(2).

Basis$|-1\rangle, |0\rangle |+1\rangle$ (labeled by σ_z eigenvalues). Reduced observable space $\sigma_x, \sigma_y, \sigma_z$. Purity = $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2$. $|-1\rangle, |1\rangle$ have pure reduced states (are generalized unentangled). $|0\rangle$ has maximally mixed reduced state.

- The unentangled pure states are precisely the "spin-coherent states."
- Exemplifies some concepts of semisimple Lie-algebra representations:

Cartan subalgebra (CSA): Maximal commuting subalgebra. Here, the one-dim algebra generated by σ_z.

- CSA's are conjugate under the Lie group $e^{\mathfrak{h}}$.

Here: the CSA's $\mathfrak{c}_\alpha$ are 1-d algebras, each generated by a spin component σ_α. Conjugate under the Lie group $SU(2)$ (you get from one to

Example: Spin-1

3-d representation of $\mathfrak{su}(2)$.

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• CSA's are conjugate under the Lie group $e^{\mathfrak{h}}$.
Here: the CSA's $\mathfrak{c}_\alpha$ are 1-d algebras, each generated by a spin component σ_α. Conjugate under the Lie group $SU(2)$ (you get from one to the other by rotation).

• Choose a CSA. The representation decomposes into orthogonal "weight spaces" which are eigenspaces of all elements of the given CSA.

Each space has a distinct pattern of real eigenvalues for CSA basis elements (i.e. linear functional on the CSA $\mathfrak{c}$), called a **weight**.

In our spin example, the weights are expectation values of the chosen spins σ_α, the weight spaces are $|0\rangle, |1\rangle, |-1\rangle$ (labelled by their spin component in direction α.

- Weights form a (special kind of) **convex polytope**. Extreme points (vertices) of the polytope are **highest weights** of the representation. States in (necessarily one-dimensional) weight spaces with highest weights (for some choice of CSA) are **highest weight states**. In our example: $|1\rangle, |-1\rangle$. (Polytope is the interval $[-1, 1]$.)

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- There is a continuous manifold of highest weight states: the orbit of any one such state under the Lie group (because there is a continuum of CSA's conjugate under that group, and a state is a highest weight state if it is in a highest weight space for some choice of CSA).
- Lie algebra has (many) orthonormal

Example: BCS

N sites/modes. Fermion creation operators $c_k^\dagger$.

Representation of $\mathfrak{u}(N)$.

$\exp \mathfrak{u}(N) = U(N)$, transformation to new modes $d_l^\dagger \to \sum_k u_{lk} c_k^\dagger$.

BCS Hamiltonian $\in \mathfrak{so}(2N)$ (includes operators $c_k c_l$, $c_k^\dagger c_l^\dagger$); ground state always generalized unentangled for $\mathfrak{so}(2N)$.

Symmetrization of state with definite numbers of Cooper pairs per mode $(\mathbf{k}, -\mathbf{k})$ available to pairs.

Mean field theory.

$d_l^\dagger \to \sum_k u_{lk} c_k$.

BCS Hamiltonian $\in SO(2N)$ (includes ... generalized

$c_k c_l$, $c_k^\dagger c_l^\dagger$); ground state always generalized unentangled for $SO(2N)$.

Symmetrization of state with definite numbers of Cooper pairs per mode $(\mathbf{k}, -\mathbf{k})$ available to pairs.

Mean field theory.

→ Basis: $\{ c_i^\dagger c_i \, , \, c_i^\dagger c_j \}$

"like" $|i\rangle\langle i|$ $\quad$ $|i\rangle\langle j|$

Example: BCS

N sites/modes. Fermion creation operators $c_k^\dagger$.

Representation of $\mathfrak{u}(N)$. $c_i^\dagger c_i$, $c_i^\dagger c_j$ $(+ c_j^\dagger c_i)$

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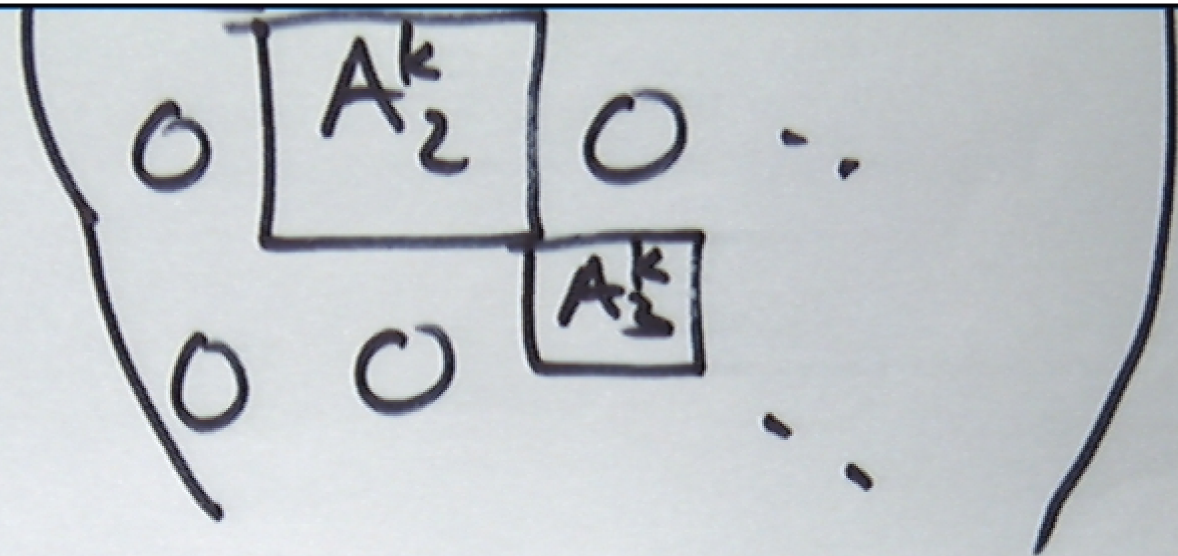
BCS Hamiltonian $\in$ so [redacted]$(2N)$ (includes operators $c_k c_l$, $c_k^\dagger c_l^\dagger$); ground state always generalized unentangled for [redacted] so $(2N)$.

Symmetrization of state with definite numbers of Cooper pairs per mode $(\mathbf{k}, -\mathbf{k})$ available to pairs.

Mean field theory.



$$|\psi\rangle = \sum_i \sqrt{p_i}\,|\psi_i\rangle$$

$$P_a(|\psi\rangle) \equiv \sum_k \left|\langle\psi_a|A^k|\psi\rangle\right|^2$$

$$= \sum_k \sum_{ij} p_i p_j \underbrace{\langle\psi_i|A_i^k|\psi_i\rangle}_{(\vec{a}_i)_k}\langle\psi_j|A_j^k|\psi_j\rangle$$

Reducible Reps (Parity)

$$A^k = \begin{pmatrix} A_1^k & 0 & 0 & \cdots \\ 0 & A_2^k & 0 & \cdots \\ 0 & 0 & A_3^k & \\ & & & \ddots \end{pmatrix}$$

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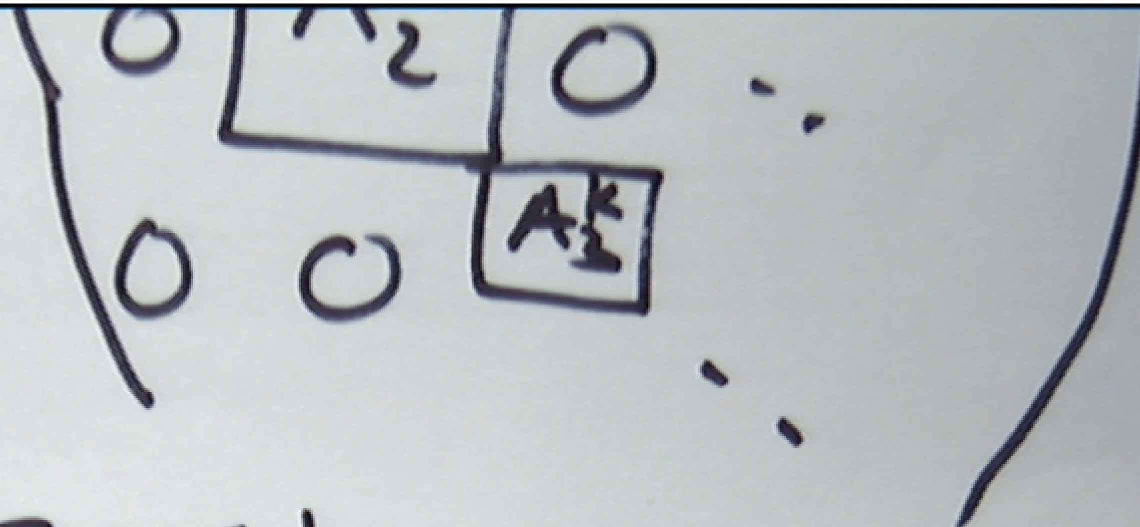

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$$= \sum p_i p_j\, \vec{a}_i \cdot \vec{a}$$

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$$= \sum_{ij} p_i p_j\, \vec{a}_i \cdot \vec{a}_k \leq \max_{i,k} \vec{a}_i \cdot \vec{a}_k$$

$$= \max |\vec{a}_i|^2 .$$

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$$= \sum_{ij} p_i p_j \, \vec{a}_i \cdot \vec{a}_k \leq \max_{i,k} \vec{a}_i \cdot \vec{a}_k$$

$$= \max_i |\vec{a}_i|^2.$$

Max $P_a \Rightarrow$ extremal. $\Leftarrow$

~~$= \rho(u+1)$~~

u(N) subalgebra: irreps are fermion-# sectors

$$P_{u(N)}\left(|\psi_{BCS}\rangle\right) = 1 - \frac{2}{N}\left(\langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2\right)$$

(special for these states)

Anisotropic XY model

$$H = \sum_{i=1}^{N}\left[(1+\gamma)\sigma_x^i\sigma_x^{i+1} + (1-\gamma)\sigma_y^i\sigma_y^{i+1} + g\sigma_z^i\right]$$

1-d lattice sites

BCS ground-states:

$H \in so(2N)$

~~[illegible]~~

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1-d lattice sites ← $i=1$

(def: $\sigma_\alpha^{i+1} := \sigma_\alpha^i$)

J.-W. transform to spinless fermions

$\frac{\cdot}{N}\left(\langle \hat{n}^2\rangle - \langle \hat{n}\rangle^2\right)$

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Anisotropic XY model

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1-d lattice sites

(def: $\sigma_\alpha^{i+1} := \sigma_\alpha^i$)

J.-W. transform to spinless fermions ...

→ BCS Hamiltonian

Jordan-Wigner Transf.

$$c_j^\dagger := \prod_{l=1}^{j-1} (-\sigma_z^l)\, \sigma_+^j$$

aniso X-Y ⟶

$$H = -2g \sum_{i=1}^{N-1} \left[\left(c_i^\dagger c_{i+1} + \gamma c_i^\dagger c_{i+1}^\dagger + h.c. \right) \right.$$

$$+ 2gK \left(c_N^\dagger c_1 + \gamma c_N^\dagger c_1^\dagger + h.c. \right)$$

$$\ell=1$$

aniso X-Y $\longrightarrow$

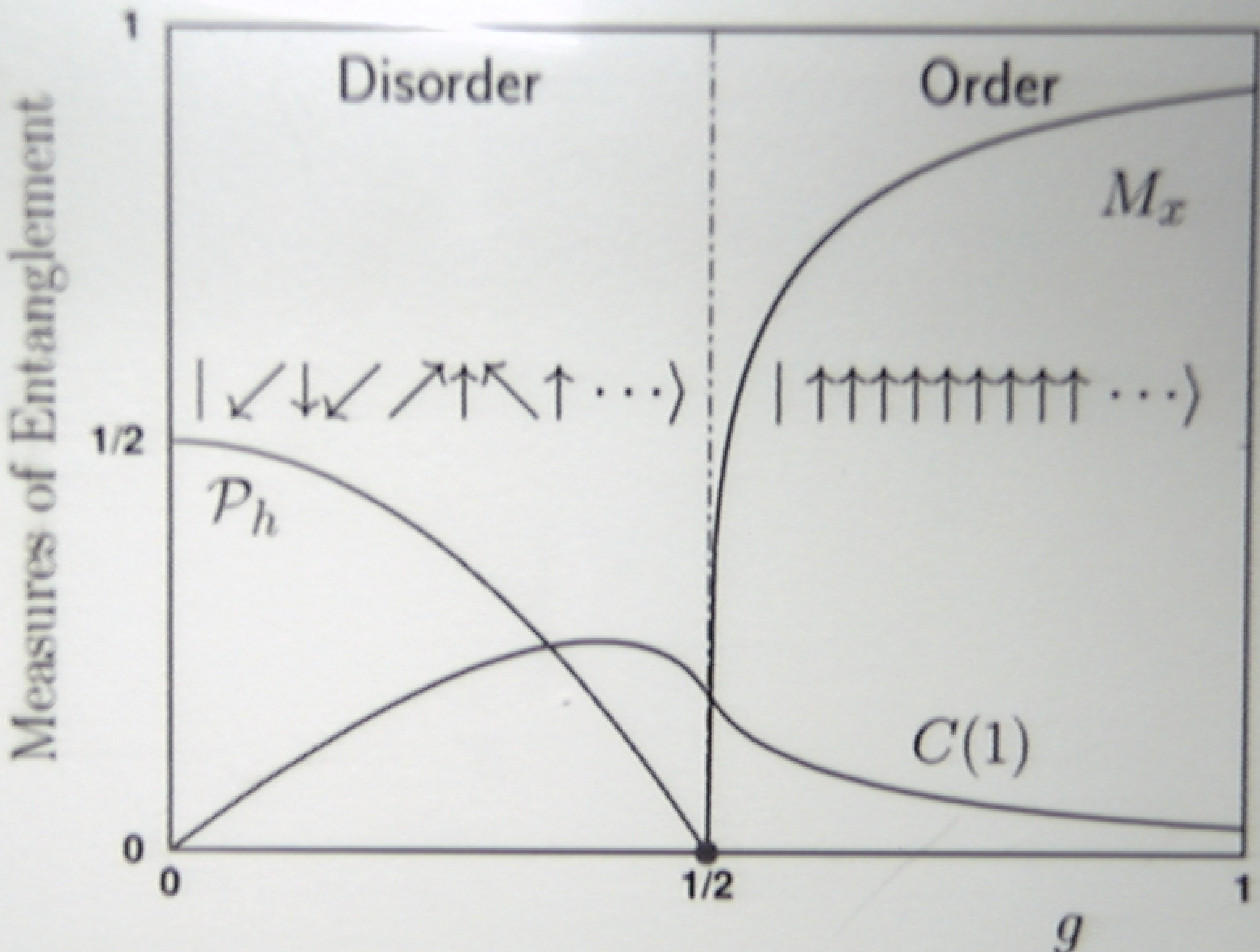
$$H = -2g \sum_{i=1}^{N-1} \Big[\left(c_i^{+} c_{i+1} + \gamma c_i^{+} c_{i+1}^{+} + h.c. \right)$$

$$+ 2gK \left(c_N^{+} c_1 + \gamma c_N^{+} c_1^{+} + h.c. \right)$$

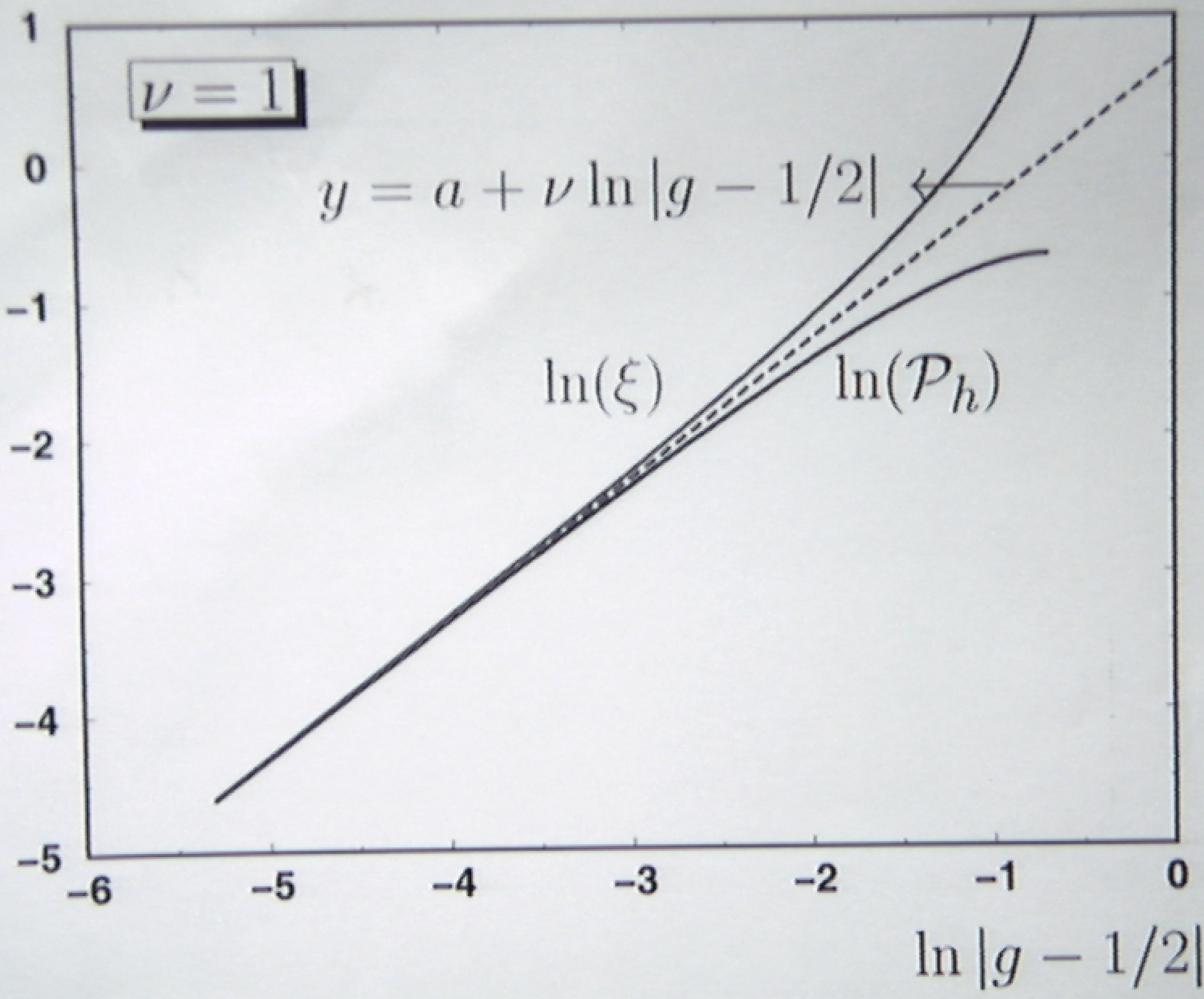
$$+ 2\hat{n}$$

where

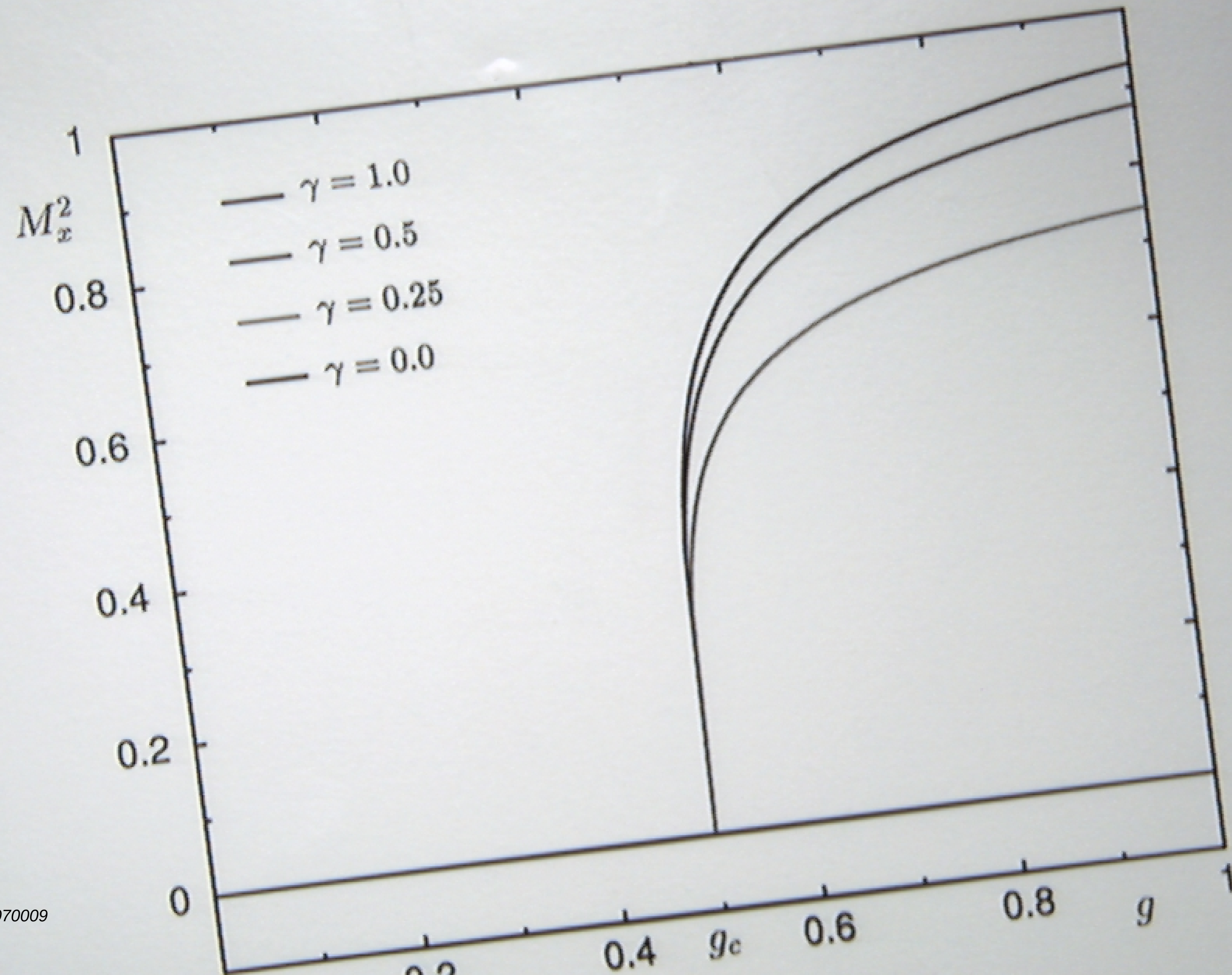
$$\hat{n} \equiv \sum_{i=1}^{N} c_i^{+} c_i \qquad K = \prod_j \left(-\sigma_z^j \right)$$



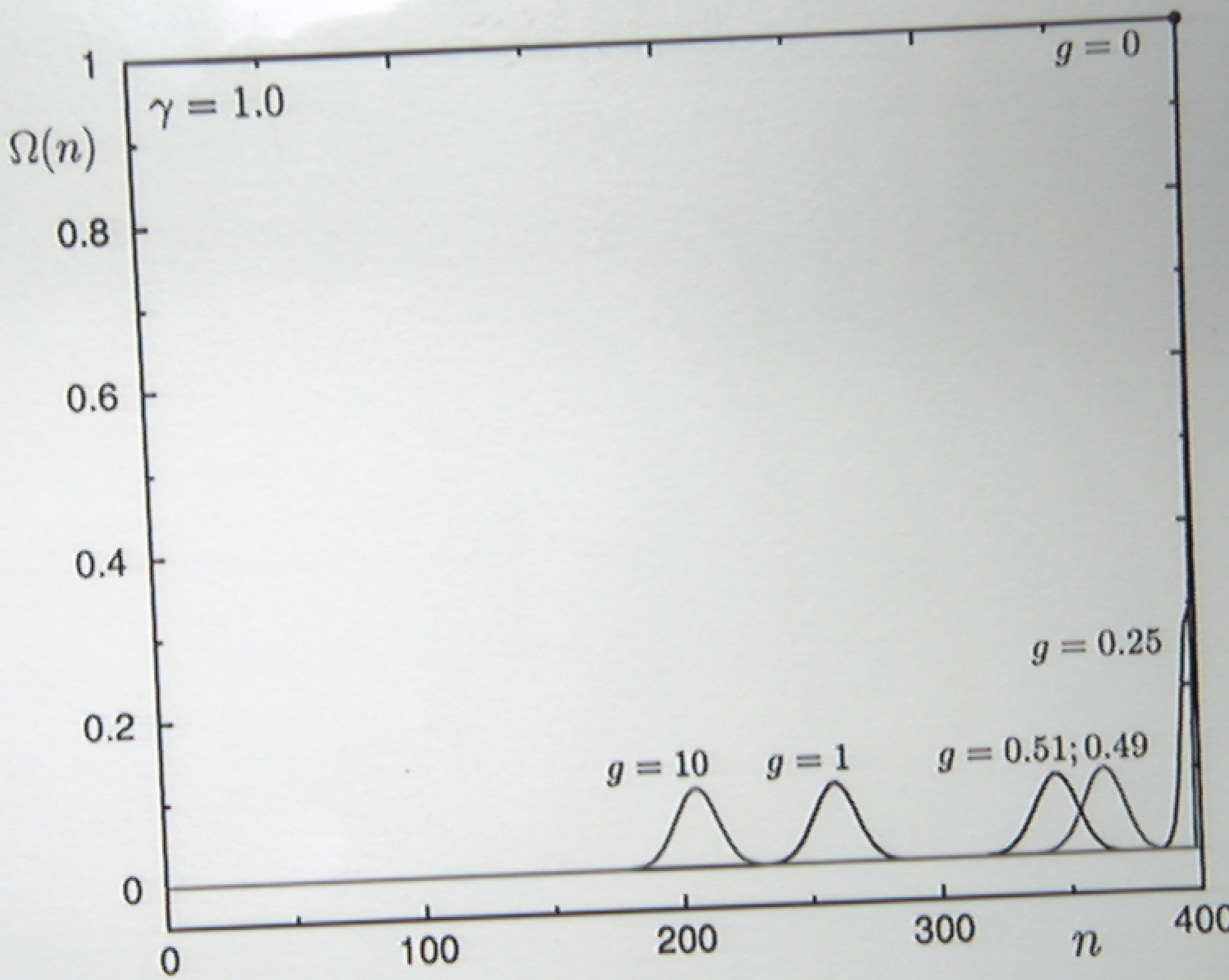
$$H = \sum_i (1+\gamma)\sigma_x^i \sigma_x^{i+1} + (1-\gamma)\sigma_y^i \sigma_y^{i+1} + g\sigma_z^i$$



$\nu = 1$
$y = a + \nu \ln |g - 1/2|$
$\ln(\xi)$
$\ln(\mathcal{P}_h)$
1
0
−1
−2
−3
−4
−5
−6
−5
−4
−3
−2
−1
0
$\ln |g - 1/2|$



1
M_x^2
0.8
0.6
0.4
0.2
0
$\gamma = 1.0$
$\gamma = 0.5$
$\gamma = 0.25$
$\gamma = 0.0$
0.4
g_c
0.6
0.8
g
1



$\gamma = 1.0$
$\Omega(n)$
1
0.8
0.6
0.4
0.2
0
$g = 0$
$g = 0.25$
$g = 10$
$g = 1$
$g = 0.51; 0.49$
0
100
200
300
400
n